15 points	30 points	30 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 Calculus I 04.12.2021

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MIDTERM EXAM

Name: ....#KEY#..

Student No: ....

1. Find the nonzero real number a such that the function

$$f(x) = \begin{cases} \dfrac{\sin ax}{x} + \cos x, & \text{if } x < 0 \\ x^2 + 3\sqrt{x+1} & \text{if } x \geq 0 \end{cases}$$

is continuous for all x.

$f(x)$ is continuous at $x=0$ if $\lim\limits_{x \to 0} f(x) = f(0)$.

- $f(0) = 0^2 + 3\sqrt{0+1} = 3$

- $$\lim_{\substack{x \to 0^- \\ x<0}} \left[\frac{\sin ax}{x} + \cos x\right] = \lim_{x \to 0^-} \left[\frac{a \cdot \sin ax}{a \cdot x} + \cos x\right]$$

$$= a \lim_{x \to 0^-} \frac{\sin ax}{ax} + \lim_{x \to 0^-} \cos x$$

$$= a + 1.$$

- $$\lim_{\substack{x \to 0^+ \\ x>0}} \left[x^2 + 3\sqrt{x+1}\right] = 3$$

$\Rightarrow \quad a+1 = 3 \quad \Rightarrow \quad a = 2.$

2. (a) Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{\sin(x^2)}{\tan x}}$.

$$y = \left(\frac{\sin(x^2)}{\tan x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\sin(x^2)}{\tan x}\right)^{-1/2} \cdot \left(\frac{\cos(x^2).2x.\tan x - \sin(x^2)\sec^2 x}{\tan^2 x}\right)$$

(b) Find $F'(4)$ if $F(x) = f(3 + 2\sqrt{x})$ and $f'(7) = 1$.

$$F'(x) = f'(3+2\sqrt{x}).(3+2\sqrt{x})'$$

$$F'(x) = f'(3+2\sqrt{x}). \frac{1}{\sqrt{x}}$$

if $x=4 \Rightarrow F'(4) = f'(3+2\sqrt{4}). \frac{1}{\sqrt{4}}$

$$= f'(7).\frac{1}{2} = 1.\frac{1}{2} = \frac{1}{2} \Rightarrow F'(4) = \frac{1}{2}.$$

(c) Find the n-th derivative $f^{(n)}(x)$ if $f(x) = \frac{1}{2-3x}$.

$$f(x) = \frac{1}{2-3x} = (2-3x)^{-1} \Rightarrow f'(x) = (-1).(2-3x)^{-2}.(-3)$$

$$f'(x) = 3.(2-3x)^{-2}$$

$$f''(x) = 3.(-2).(2-3x)^{-3}.(-3)$$

$$f''(x) = 2.3^2.(2-3x)^{-3}$$

$$f'''(x) = 2.3^2.(-3).(2-3x)^{-4}.(-3)$$

$$f'''(x) = 3^3.2.3.(2-3x)^{-4}$$

$f^{(n)}(x) = n!.3^n.(2-3x)^{-(n+1)}$, where n is a positive integer.

3. Evaluate the following limit or explain why it does not exist.

(a) $\lim_{x\to 2} \dfrac{x^2-4}{3x|x-2|}$

x		2	
$x-2$	$-$	0	$+$

$$\lim_{\substack{x\to 2^- \\ x<2}} \frac{\cancel{(x-2)}(x+2)}{3x\cdot\cancel{(-x+2)}} = \lim_{x\to 2^-} \frac{-(x+2)}{3x} = \frac{-4}{6} = \frac{-2}{3}.$$

$\neq$

$$\lim_{\substack{x\to 2^+ \\ x>2}} \frac{\cancel{(x-2)}(x+2)}{3x\cancel{(x-2)}} = \lim_{x\to 2^+} \frac{x+2}{3x} = \frac{4}{6} = \frac{2}{3}$$

Since, the left limit is not equal to the right limit, the limit does not exist!

(b) $\lim_{x\to 3} \dfrac{2x^2-1}{(x-3)^5}$

		$-1/\sqrt{2}$		$1/\sqrt{2}$		3	
$2x^2-1$	$+$	0	$-$	0	$+$		$+$
$(x-3)^5$	$-$		$-$		$-$	0	$+$
$f(x)$	$-$		$+$		$-$		$+$

$$\lim_{\substack{x\to 3^- \\ x<3}} \frac{2x^2-1}{(x-3)^5} = -\infty$$

$\neq \Rightarrow$ the limit does not exist!

$$\lim_{x\to 3^+} \frac{2x^2-1}{(x-3)^5} = +\infty$$

(c) $\lim_{x\to -\infty} (x+\sqrt{x^2-2x+3})$

$$\lim_{x\to -\infty} \frac{(x+\sqrt{x^2-2x+3})(x-\sqrt{x^2-2x+3})}{(x-\sqrt{x^2-2x+3})} = \lim_{x\to -\infty} \frac{\cancel{x^2}-\cancel{x^2}+2x-3}{x-\sqrt{x^2-2x+3}}$$

$$= \lim_{x\to -\infty} \frac{2x-3}{x-|x|\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}} = \lim_{x\to -\infty} \frac{\cancel{x}\cdot\left(2-\frac{3}{x}\right)}{\cancel{x}\cdot\left[1+\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}\right]} = \frac{2}{2}$$

$|x| = -x$

$x<0$

$= 1.$

So, the limit is 1.

4. Find the equation of the tangent and normal lines to the graph of

$$\sin\left(\frac{\pi x}{y}\right) = \frac{y^2}{x} - 3$$

at the point $(1, 2)$.

$$\cos\left(\frac{\pi x}{y}\right) \cdot \left(\frac{\pi . y - \pi . x . y'}{y^2}\right) = \frac{2y . y' . x - y^2 . 1}{x^2}$$

At the point $(1,2)$:

$$\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} \cdot \left(\frac{2\pi - \pi . y'}{2^2}\right) = \frac{2 . 2 . y' - 4}{1} \Rightarrow 4y' = 4$$

$$y' = 1.$$

So, the slope of the tangent line $\boxed{m_T = 1.}$ at $(1,2)$.

The tangent line has the equation:

$$y - 2 = m_T(x-1) \Rightarrow y = (x-1) + 2 \Rightarrow \boxed{y = x + 1.}$$

Since $m_T \cdot m_N = -1 \Rightarrow$ Normal line has the slope $\boxed{m_N = -1}$ at $(1,2)$.

Its equation is

$$y - 2 = m_N \cdot (x-1) \Rightarrow y = (-1)(x-1) + 2$$

$$\boxed{y = -x + 3.}$$