15 points	35 points	30 points	20 points	100 points
1	2	3	4	**Total**

MATH 153 Calculus I

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MIDTERM EXAM

Name: ....#KEY#..........................

Student No: ....

I. Find the nonzero real number a such that the function

$$f(x) = \begin{cases} \dfrac{x\cos(x)}{\sin(ax)}, & \text{if } x < 0 \\ (2-x)^2 - 6, & \text{if } x \geq 0 \end{cases}$$

is continuous at $x = 0$.

$$\lim_{x \to 0} f(x) = f(0)$$

$$f(0) = 2^2 - 6 = -2$$

$$\lim_{\substack{x \to 0^+ \\ x > 0}} (2-x)^2 - 6 = -2$$

$$\lim_{\substack{x \to 0^- \\ x < 0}} \frac{x\cos x}{\sin(ax)} = \lim_{x \to 0^-} \frac{a.x}{\sin(ax)} \cdot \frac{1}{a} \cdot \cos x = \frac{1}{a}$$

$$\text{so } \frac{1}{a} = -2 \implies a = \frac{1}{-2}$$

$$* \lim_{x \to 0^-} \frac{a.x}{\sin ax} = \lim_{x \to 0^-} \frac{1}{\frac{\sin(ax)}{ax}} = 1.$$

2. Evaluate the following limit or explain why it does not exist.

(a) $\lim_{x\to 1} \frac{|x-1|}{|x|-1}$

		0		$\rightarrow 1 \leftarrow$	
$x-1$	$-$		$-$	0	$+$
x	$-$	0	$+$		$+$

$$\lim_{\substack{x\to 1^- \\ x<1}} \frac{-\cancel{(x-1)}}{\cancel{x-1}} = -1$$

$\neq$ so limit does not exist.

$$\lim_{\substack{x\to 1^+ \\ x>1}} \frac{\cancel{x-1}}{\cancel{x-1}} = 1$$

(b) $\lim_{x\to 5} \frac{x^2-4}{(x-5)^3}$: inf. limit

		-2		2		$\rightarrow 5 \leftarrow$	
x^2-4	$+$	0	$-$	0	$+$		$+$
$(x-5)^3$	$-$		$-$		$-$	0	$+$

$$\lim_{\substack{x\to 5^- \\ x<5}} \frac{\overset{+}{x^2-4}}{\underset{-}{(x-5)^3}} = -\infty$$

$\neq$ so limit does not exist.

$$\lim_{x\to 5^+} \frac{\overset{+}{x^2-4}}{\underset{+}{(x-5)^3}} = +\infty$$

(c) $\lim_{x\to\infty} (\sqrt{2x+4x^2} - 2x)$: $(\infty - \infty)$

$$\lim_{x\to\infty} \frac{(\sqrt{2x+4x^2}-2x)(\sqrt{2x+4x^2}+2x)}{(\sqrt{2x+4x^2}+2x)} = \lim_{x\to\infty} \frac{2x+\cancel{4x^2}-\cancel{4x^2}}{(\sqrt{2x+4x^2}+2x)}$$

$$= \lim_{\substack{x\to\infty \\ |x|=x}} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt{\frac{2}{x}+4}+2\right)} = \lim_{x\to\infty} \frac{2}{\sqrt{\frac{2}{x}+4}+2} = \frac{2}{2+2} = \frac{1}{2}.$$

3. (a) Find $f'(x)$ if $f(x) = \dfrac{1}{(7+\sqrt{3x})^{\frac{2}{3}}}$ $\iff f(x) = (7+\sqrt{3x})^{-\frac{2}{3}}$

$$f'(x) = \left(-\frac{2}{\cancel{3}}\right) \cdot (7+\sqrt{3x})^{-\frac{5}{3}} \cdot \frac{1}{\cancel{2}\sqrt{3x}} \cdot \cancel{3}$$

so

$$f'(x) = -(7+\sqrt{3x})^{-\frac{5}{3}} \cdot \frac{1}{\sqrt{3x}} \quad \text{OR} \quad f'(x) = \frac{-1}{\sqrt{3x}\,(7+\sqrt{3x})^{5/3}}$$

(b) Find $F'(\frac{\pi}{2})$ if $F(x) = f(\pi + \cos x)$ and $f'(\pi) = -1$.

$$F'(x) = f'(\pi + \cos x) \cdot (-\sin x)$$

if $x = \frac{\pi}{2}$, $F'\left(\frac{\pi}{2}\right) = f'\left(\pi + \cos\frac{\pi}{2}\right) \cdot \left(-\sin\frac{\pi}{2}\right)$

$$= f'(\pi) \cdot \left(-\sin\frac{\pi}{2}\right)$$

$$= (-1) \cdot (-1)$$

$$= 1.$$

(c) Find $f'(x)$ if $f(x) = \tan^2(1+x^3)$.

$$f'(x) = 2\tan(1+x^3) \cdot \sec^2(1+x^3) \cdot 3x^2.$$

4. **Find the equations of the tangent and normal lines to the curve**

$$\frac{x}{y} - x^2y = \sin(x^2 + y)$$

at the point $(0, \pi)$.

$$\frac{1.y - x.y'}{y^2} - 2xy - x^2.y' = \cos(x^2+y).(2x+y')$$

if $x=0$, $y=\pi$;

$$\frac{1.\pi - \overbrace{0.y'}^{0}}{\pi^2} - \overbrace{2.0.\pi}^{0} - \overbrace{0^2.y'}^{0} = \cos(\pi).(2.0+y')$$

$$\frac{1}{\pi} = -y' \Rightarrow y' = \frac{-1}{\pi}$$

Slope of tangent line: $m_T = \frac{-1}{\pi}$

Slope of normal line: $m_N \cdot \left(-\frac{1}{\pi}\right) = -1 \Rightarrow m_N = \pi$

Equation of tangent line: $y - \pi = \left(-\frac{1}{\pi}\right)x$

OR : $y = -\frac{x}{\pi} + \pi$

Equation of normal line: $y - \pi = \pi.x$

OR: $y = \pi.x + \pi$