20 points	25 points	30 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 - CALCULUS I 25.11.2023

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MIDTERM

Name and Surname: ....

Student Number: ....

1. Find an equation of the tangent line to the curve

$$\frac{1}{(x-1)^2+y^2}+\frac{1}{x^2+(y-x)^2}=2$$

at the point $(1,1)$.

$$-\frac{2.(x-1)+2yy'}{[(x-1)^2+y^2]^2}-\frac{2x+2(y-x)(y'-1)}{[x^2+(y-x)^2]^2}=0$$

At the point $(1,1)$ we obtain

$$m=y'\Big|_{(1,1)}=-1.$$

Then we have the equation of the tangent line

$$y=m(x-x_0)+y_0$$

$$y=-1.(x-1)+1$$

$$y=-x+2$$

2. (a) **Use the definition of derivative** to find $f'(x)$ for the function $f(x) = \frac{x}{x-3}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h-3} - \frac{x}{x-3}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x-3) - x(x+h-3)}{h(x-3)(x+h-3)} = \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(x-3)(x+h-3)}$$

$$= \lim_{h \to 0} \frac{-3}{(x-3)(x+h-3)} = \frac{-3}{(x-3)^2}$$

(b) Let

$$f(x) = \begin{cases} \dfrac{\tan(6x)}{\tan(3x)}, & \text{if } x \neq 0 \\ 2k - 1, & \text{if } x = 0 \end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 0$.

If $f(x)$ is continuous at $x=0$, we have

$$\lim_{x \to 0} f(x) = f(0).$$

$f(0) = 2k-1$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\tan(6x)}{\tan(3x)} = \lim_{x \to 0} \frac{\frac{\sin(6x)}{\cos(6x)}}{\frac{\sin(3x)}{\cos(3x)}}$$

$$= \underbrace{\lim_{x \to 0} \frac{\sin 6x}{6x}}_{1} \cdot 6 \cdot \underbrace{\lim_{x \to 0} \frac{3x}{\sin 3x}}_{1} \cdot \frac{1}{3} \cdot \underbrace{\lim_{x \to 0} \frac{\cos(3x)}{\cos(6x)}}_{1}$$

$= 2$

$2k - 1 = 2 \Rightarrow k = \frac{3}{2}$

3. Calculate the limit or explain why it does not exist (**Do not use L'Hôpital's Rule**)

(a) $\lim_{x\to 0}(\frac{1}{x\sqrt{4+x}} - \frac{1}{2x})$

$$\lim_{x\to 0}\left[\frac{1}{x\sqrt{4+x}} - \frac{1}{2x}\right] = \lim_{x\to 0}\frac{2-\sqrt{4+x}}{2x\sqrt{4+x}}$$

$$= \lim_{x\to 0}\frac{(2-\sqrt{4+x}).(2+\sqrt{4+x})}{2x\sqrt{4+x}.(2+\sqrt{4+x})} = \lim_{x\to 0}\frac{-x}{2x\sqrt{4+x}\,(2+\sqrt{4+x})}$$

$$= \lim_{x\to 0}\frac{-1}{2\sqrt{4+x}\,(2+\sqrt{4+x})} = \frac{-1}{16}$$

(b) $\lim_{x\to 3^-}\frac{\frac{1}{|x-3|} - \frac{1}{x-3}}{x-3}$

x	3
$x-3$	$-$ ⦶ $+$

$$\lim_{x\to 3^-}\frac{-\frac{1}{x-3} - \frac{1}{x-3}}{x-3} = \lim_{x\to 3^-}\frac{-2}{(x-3)^2}$$

$$= -\infty$$

(c) $\lim_{x\to -\infty}\frac{\sqrt{4x^2+x+1}}{3x-1}$

$$\lim_{x\to -\infty}\frac{\sqrt{4x^2+x+1}}{3x-1} = \lim_{x\to -\infty}\frac{|x|.\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}}{x.(3-\frac{1}{x})}$$

$$= \lim_{x\to -\infty}\frac{-x\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}}{x.(3-\frac{1}{x})} = \lim_{x\to -\infty}\frac{-\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}}{3-\frac{1}{x}} = \frac{-2}{3}$$

since $\lim_{x\to -\infty}\frac{1}{x} = 0$ and $\lim_{x\to -\infty}\frac{1}{x^2} = 0$.

4. (a) Find $\frac{dy}{dx}$ if $y = \cos^3(\sin(\tan\ 5x))$.

$y = [\cos(\sin(\tan 5x))]^3$

$y' = 3 \cdot [\cos(\sin(\tan 5x))]^2 \cdot [\cos(\sin(\tan 5x))]'$

$= 3 \cdot \cos^2(\sin(\tan 5x)) \cdot [-\sin(\sin(\tan 5x))] \cdot$

$[\sin(\tan 5x)]'$

$= -3 \cdot \cos^2(\sin(\tan 5x)) \cdot \sin(\sin(\tan 5x)) \cdot \cos(\tan 5x) \cdot$

$5 \cdot \sec^2 5x$

$= -15 \cdot \cos^2(\sin(\tan 5x)) \cdot \sin(\sin(\tan 5x)) \cdot \cos(\tan 5x)$

$\sec^2 5x$

(b) Suppose that f is a differentiable $f(1) = 1$, $f(2) = -1$, $f'(1) = 1$ and $f'(2) = 3$.

If $F(x) = \left[f\left(\frac{1}{x} + f(\frac{1}{x^2})\right)\right]^2$, then find $F'(1)$.

$F'(x) = 2 \cdot [f(\frac{1}{x} + f(\frac{1}{x^2}))] \cdot [f(\frac{1}{x} + f(\frac{1}{x^2}))]'$

$= 2 \cdot f(\frac{1}{x} + f(\frac{1}{x^2})) \cdot f'(\frac{1}{x} + f(\frac{1}{x^2})) \cdot [\frac{1}{x} + f(\frac{1}{x^2})]'$

$= 2 \cdot f(\frac{1}{x} + f(\frac{1}{x^2})) \cdot f'(\frac{1}{x} + f(\frac{1}{x^2})) \cdot [-\frac{1}{x^2} - \frac{3}{x^4} f'(\frac{1}{x^2})$

$F'(1) = 2 \cdot f(\underbrace{1 + \underbrace{f(1)}_{1}}_{2}) \cdot f'(\underbrace{1 + \underbrace{f(1)}_{1}}_{2}) \cdot [-1 - 3 \cdot \underbrace{f'(1)}_{1}]$

$= 2 \cdot (-1) \cdot 3 \cdot (-4)$

$= 24$