KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 02.01.2020

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FINAL EXAM

Student Name:...

Student Number:...

1. (a) Consider the following piecewise defined function

$$f(x) = \begin{cases} x+4 & ; x < 1 \\ ax^2 + bx + 2 & ; 1 \leq x < 3 \\ 6x + a - b & ; x \geq 3 \end{cases}$$

Find the constants a and b so that f is continuous at both $x = 1$ and $x = 3$.

$x = 1$:

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+4) = 5$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2 + bx + 2) = a + b + 2$

$f(1) = a + b + 2$

$a + b + 2 = 5 \Rightarrow \boxed{a + b = 3}$

$x = 3$:

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 + bx + 2) = 9a + 3b + 2$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + a - b) = 18 + a - b$

$f(3) = 18 + a - b$

$18 + a - b = 9a + 3b + 2 \Rightarrow \boxed{8a + 4b = 16}$

$a + b = 3 \Rightarrow b = 3 - a \Rightarrow 8a + 4(3 - a) = 16 \Rightarrow a = 1$

$\Downarrow$

$b = 2$

(b) Find an equation of the line tangent to the graph of $x^2 + (y - x)^3 = 9$ at $x = 1$.

when $x = 1 \Rightarrow 1 + (y-1)^2 = 9 \Rightarrow (y-1)^2 = 8 \Rightarrow y = 3$

Find the derivative:

$2x + 3(y - x)^2 \cdot (y' - 1) = 0$

Evaluate derivative at $(1, 3)$:

$2 + 3(3-1)^2 \cdot (y' - 1) = 0 \Rightarrow y' = \frac{5}{6}$

Equation of tangent line:

$y - 3 = \frac{5}{6}(x - 1)$

2. (a) Find the absolute maximum and minimum values of $f(x) = x^3 + \frac{48}{x}$ on $[1, 4]$.

$f'(x) = 3x^2 - \frac{48}{x^2} = \frac{3x^4 - 48}{x^2} = 0 \Rightarrow x^4 = 16 \Rightarrow x = \mp 2$

$x = 1 \Rightarrow f(1) = 1 + \frac{48}{1} = 49$

$x = 2 \Rightarrow f(2) = 8 + \frac{48}{2} = 32$ absolute min

$x = 4 \Rightarrow f(4) = 64 + \frac{48}{4} = 76$ absolute max

(b) The area of a circle is decreasing at a rate of $3cm^2/min$. How fast is the radius of the circle changing when the area is $300cm^2$.

$A = \pi r^2$

$\frac{dA}{dt} = -3 cm^2/min$

$A = 300 cm^2$

$\frac{dr}{dt} = ?$

$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$

$-3 = \pi \cdot 2 \cdot \sqrt{\frac{300}{\pi}} \cdot \frac{dr}{dt}$

when $A = 300 cm^2$

$r^2 = \frac{300}{\pi} \Rightarrow r = \sqrt{\frac{300}{\pi}}$

$\frac{dr}{dt} = \frac{-3}{2\sqrt{\pi} \cdot \sqrt{300}}$ cm/min

(c) Find two nonnegative numbers x and y for which $2x + y = 30$ such that xy^2 is maximized.

$x, y \geq 0$ $\quad 2x + y = 30 \Rightarrow y = 30 - 2x$; $x \in [0, 15]$

$xy^2 = x(30-2x)^2$

Find critical points of $x(30-2x)^2$:

$[x(30-2x)^2]' = (30-2x)^2 + x \cdot 2 \cdot (30-2x) \cdot (-2) = 0$

$\Rightarrow (30-2x)[30-2x-4x] = 0 \Rightarrow (30-2x)(30-6x) = 0$

$x = 15$; $x = 5$

$\Downarrow \quad \Downarrow$

$y = 0 \quad y = 20$

$xy^2\Big|_{(15,0)} = 0$; $xy^2\Big|_{(5,20)} = 5 \cdot (20)^2$ abs. max

3. (a) Find the vertical and horizantal asymptotes of the given function

$$f(x) = \frac{2x^2}{x^2-4}$$

$\lim_{x\to\infty} \frac{2x^2}{x^2-4} = \lim_{x\to\infty} \frac{2}{1-\frac{4}{x^2}} = 2$; $\lim_{x\to-\infty} \frac{2x^2}{x^2-4} = \lim_{x\to-\infty} \frac{2}{1-\frac{4}{x^2}} = 2$

So $y=2$ is the horizontal asymptote.

$\lim_{x\to 2^-} \frac{2x^2}{x^2-4} = -\infty$, $\lim_{x\to 2^+} \frac{2x^2}{x^2-4} = \infty$ } $x=2$ is a vertical asymptote

$\lim_{x\to -2^-} \frac{2x^2}{x^2-4} = \infty$, $\lim_{x\to -2^+} \frac{2x^2}{x^2-4} = -\infty$ } $x=-2$ is a vertical asymptote

(b) Evaluate $\lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}$

Let $y = (x+e^x)^{\frac{2}{x}}$ then $\ln y = \ln (x+e^x)^{\frac{2}{x}}$

$\ln y = \frac{2}{x} \ln (x+e^x)$

$\lim_{x\to\infty} \frac{2\ln(x+e^x)}{x} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{2\frac{1}{x+e^x}\cdot(1+e^x)}{1}$

$= \lim_{x\to\infty} \frac{2(1+e^x)}{x+e^x} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{2e^x}{1+e^x} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{2e^x}{e^x} = 2$

$\lim_{x\to\infty} \ln y = 2 \Rightarrow$

$\lim_{x\to\infty} y = \lim_{x\to\infty} (x+e^x)^{\frac{2}{x}} = e^2$

4. (a) Evaluate the following integrals

i. $\int_0^1 x^2(1+2x^3)^5 dx$

Let $t = 1+2x^3 \Rightarrow \frac{dt}{dx} = 6x^2 \Rightarrow \frac{1}{6}dt = x^2 dx$

$x=0 \Rightarrow t=1$

$x=1 \Rightarrow t=3$

$$= \int_1^3 t^5 \cdot \frac{1}{6} dt = \frac{1}{6} \cdot \frac{t^6}{6}\Big|_1^3 = \frac{1}{36}(3^6-1)$$

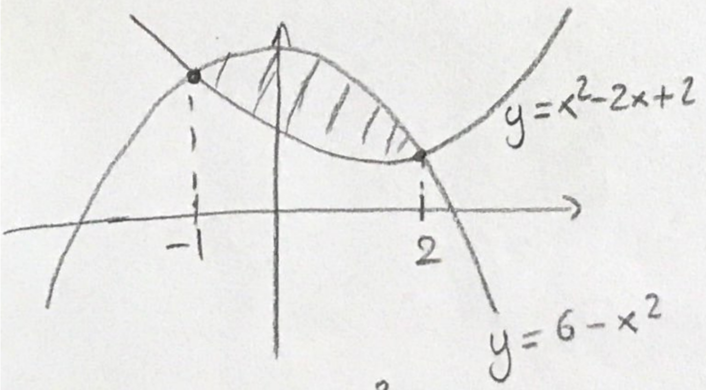
ii. $\int \frac{sinx}{1-cosx} dx$

Let $u = 1-\cos x$ then $du = \sin x \, dx$

$$= \int \frac{1}{u} du = \ln|u| + c = \ln|1-\cos x| + c$$

(b) Sketch and find the area of the plane region bounded by the curves

$$y = x^2 - 2x + 2 \quad \text{and} \quad y = 6 - x^2$$



Intersection points:

$x^2-2x+2 = 6-x^2$

$2x^2-2x-4=0$

$x^2-x-2=0$

$(x-2)(x+1)=0 \Rightarrow x=2;\ x=-1$

$$\text{Area} = \int_{-1}^{2} \left[(6-x^2)-(x^2-2x+2)\right] = \int_{-1}^{2} \left[4+2x-2x^2\right] dx$$

$$= \left(4x + \frac{2x^2}{2} - \frac{2x^3}{3}\right)\Big|_{-1}^{2}$$