40 points	30 points	15 points	15 points	100 points
1	2	3	4	**Total**

MATH 154 - Calculus II 24.04.2022

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

MIDTERM EXAM

Name and Surname: KEY

Student Number: ....

1. **Evaluate the indicated integrals**

(a) $\int \sin x \sin(2x)dx = I$

1st Way

Integ. by parts

$u = \sin 2x \quad dv = \sin x\,dx$

$du = 2\cos 2x \quad v = -\cos x$

$= (\sin 2x).(-\cos x) + 2\int \cos x \cos 2x\,dx$

$u = \cos 2x \quad dv = \cos x\,dx$

$du = -2\sin 2x\,dx \quad v = \sin x$

2nd Way: $\sin 2x = 2\sin x \cos x$

$I = 2\int \sin^2 x \cos x\,dx$ $\quad \Big|\ \sin x = u,\ \cos x\,dx = du$

$= 2\int u^2 du$

$= \frac{2}{3}u^3 + C = \frac{2}{3}(\sin x)^3 + C.$

$$I = (\sin 2x)(-\cos x) + 2\left(\sin x \cos 2x + 2\int \sin x \sin 2x\,dx\right)$$

$$I = -\sin 2x.\cos x + 2\sin x \cos 2x + 4I$$

$$I = \left(\frac{-1}{3}\right)\left(-\sin 2x.\cos x + 2\sin x\cos 2x\right) + C$$

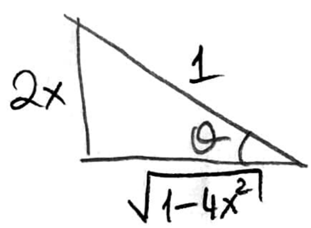
(b) $\int \frac{x+1}{\sqrt{1-4x^2}}dx$

$2x = \sin\theta \rightarrow \theta = \sin^{-1}2x$

$2dx = \cos\theta\,d\theta \rightarrow dx = \frac{1}{2}\cos\theta\,d\theta$

$\sqrt{1-4x^2} = \cos\theta$

$$= \frac{1}{2}\int \frac{\frac{1}{2}\sin\theta + 1}{\cancel{\cos\theta}}.\cancel{\cos\theta}\,d\theta = \frac{1}{2}\int\left(\frac{1}{2}\sin\theta + 1\right)d\theta = \frac{1}{2}\left(-\frac{1}{2}\cos\theta + \theta\right) + C$$



$\cos\theta = \sqrt{1-4x^2} \Longrightarrow = \frac{-1}{4}\sqrt{1-4x^2} + \frac{\sin^{-1}2x}{2} + C.$

OR

$$\int \frac{x}{\sqrt{1-4x^2}}dx + \int \frac{1}{\sqrt{1-4x^2}}dx$$

$1-4x^2 = u$

$-8x\,dx = du$

$x\,dx = -\frac{1}{8}du$

$$= -\frac{1}{8}\int \frac{du}{u^{1/2}} + \int \frac{1}{\sqrt{1-4x^2}}dx$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + \frac{1}{2}\sin^{-1}2x + C$$

(c) $\int \frac{x+3}{x^2-4x+3}dx$

$x^2 - 4x + 3 = (x-3)(x-1)$

$$\frac{x+3}{(x-3)(x-1)} = \frac{A}{x-3}\Big|_{x-1} + \frac{B}{x-1}\Big|_{x-3} = \frac{Ax - A + Bx - 3B}{(x-1)(x-3)}$$

$\cancel{A} + B = 1$

$-\cancel{A} - 3B = 3$

$-2B = 4 \rightarrow \boxed{B=-2}$

$\boxed{A=3}$

$$\int \frac{3}{x-3}dx - \int \frac{2}{x-1}dx = 3\ln(x-3) - 2\ln(x-1) + C$$

2. Evaluate the improper integral or explain why it diverges.

(a) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1-\ln x}}$

$1-\ln x = u$

$-\frac{1}{x}dx = du$

if $x=1$, $u = 1-\ln 1 = 1$

if $x=e$, $u = 1-\ln e = 0$

$$= -\int_1^0 \frac{du}{\sqrt{u}} = \lim_{R\to 0} \int_R^1 u^{-1/2}\,du = \lim_{R\to 0} \left.\frac{u^{1/2}}{\frac{1}{2}}\right|_R^1$$

$$= \lim_{R\to 0}\left(2\sqrt{1} - 2\sqrt{R}\right) = 2$$

Improper integral converges to 2.

(b) $\displaystyle\int_3^\infty \frac{dx}{(x-2)^{\frac{2}{3}}}$

$x-2 = u$

$dx = du$

If $x=3$, $u=1$

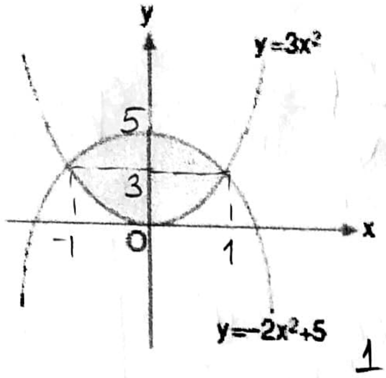
If $x\to\infty$, $u = x-2 \to \infty$

$$= \int_1^\infty \frac{du}{u^{2/3}} = \lim_{R\to\infty} \int_1^R u^{-2/3}\,du$$

$$= \lim_{R\to\infty} \left.\frac{u^{1/3}}{\frac{1}{3}}\right|_1^R = \lim_{R\to\infty}\left(3\sqrt[3]{R} - 3\sqrt[3]{1}\right) = \infty$$

Improper Integral diverges to ∞.

3. Find the volume of the solid of revolution which is obtained by rotating the below-given region between the curves $y = 3x^2$ and $y = -2x^2 + 5$ around the <u>x-axis.</u>



Intersection:

$$3x^2 = -2x^2 + 5$$

$$5x^2 = 5 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1.$$

Slicing:

$$V = \pi \int_{-1}^{1} \left[(-2x^2+5)^2 - (3x^2)^2\right] dx$$

$$V = \pi \int_{-1}^{1} (4x^4 - 20x^2 + 25 - 9x^4)\,dx = \pi \int_{-1}^{1} (-5x^4 - 20x^2 + 25)\,dx$$

$$= \pi \left(-x^5 - \frac{20}{3}x^3 + 25x\right)\Big|_{-1}^{1} = \pi\left[\left(-1 - \frac{20}{3} + 25\right) - \left(+1 + \frac{20}{3} - 25\right)\right]$$

$$= \pi\left(24 - \frac{20}{3} + 24 - \frac{20}{3}\right) = \frac{104\pi}{3}.$$

OR: cylindrical shell. (region is symmetric about y-axis)

$$V = 2\cdot\left[2\pi \int_0^3 y\left(\sqrt{\frac{y}{3}}\right)dy + 2\pi \int_3^5 y\cdot\sqrt{\frac{5}{2} - \frac{y}{2}}\,dy\right]$$

$y = -2x^2 + 5$, $\quad y = 3x^2$

$2x^2 = 5 - y$ $\quad x^2 = \frac{y}{3}$

$x^2 = \frac{5}{2} - \frac{y}{2}$ $\quad x = \sqrt{\frac{y}{3}}$

$x = \sqrt{\frac{5}{2} - \frac{y}{2}}$

$\frac{y}{3} = \vartheta$

$\frac{dy}{3} = d\vartheta$

$dy = 3\,d\vartheta$

if $y = 0$, $\vartheta = 0$

if $y = 3$, $\vartheta = 1$

$\frac{5}{2} - \frac{y}{2} = u \Rightarrow 5 - 2u = y$

$-\frac{1}{2}dy = du \rightarrow dy = -2\,du$

if $y = 3$, $u = 1$

if $y = 5$, $u = 0$

$$V = 2\cdot\left[2\pi 3 \int_0^1 3\vartheta\cdot\vartheta^{1/2}\,d\vartheta + 2\pi\cdot(-2)\int_1^0 (5-2u)\cdot u^{1/2}\,du\right]$$

$$V = 2\cdot\left[18\pi\, \frac{2}{5}\cdot\vartheta^{5/2}\Big|_0^1 + 4\pi\left(\frac{10}{3}u^{3/2} - \frac{4}{5}u^{5/2}\right)\Big|_0^1\right] = 2\cdot\left(\frac{36\pi}{5} + \frac{40\pi}{3} - \frac{16\pi}{5}\right) = \frac{104\pi}{3}$$

4. Use Maclaurin series of e^x, to find the Maclaurin series of the function

$$f(x) = 5x^2 e^{-5x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-5x^2} = \sum_{n=0}^{\infty} \frac{(-5x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n \cdot x^{2n}}{n!}$$

$$5x^2 \cdot e^{-5x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5 \cdot 5^n x^2 \cdot x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^{n+1} \cdot x^{2n+2}}{n!}$$