25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 30.05.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

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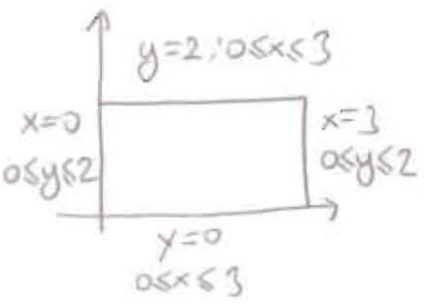
1. (a) Find the maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle: $0 \le x \le 3$, $0 \le y \le 2$.

Find the critical points: $f_x(x,y)=0$, $f_y(x,y)=0$

$f_x(x,y) = 2x-2y=0 \Rightarrow x=y$; $f_y(x,y) = -2x+2=0 \Rightarrow x=1 \Rightarrow y=1$

$\therefore$ $(1,1)$ is the only critical point, $f(1,1)=1$



$x=0 ; 0\le y\le 2$:

$f(0,y) = 2y$

$f(0,0)=0$

$f(0,2)=4$

$f_y(0,y) = 2 \neq 0$ (no critical point)

$x=3 ; 0\le y \le 2$:

$f(3,y) = 9-4y$

$f(3,0)=9$

$f(3,2)=1$

$f_y(3,y) = -4 \neq 0$ (no critical point)

$y=0 ; 0\le x\le 3$

$f(x,0)=x^2$

$f_x(x,0) = 2x = 0 \Rightarrow x=0$

$f(0,0)=0$, $f(3,0)=9$

$y=2 ; 0\le x \le 3$

$f(x,2) = x^2-4x+4$

$f_x(x,2) = 2x-4=0 \Rightarrow x=2$

$f(0,2)=4$, $f(3,2)=5$

$f(2,2)=0$

$\therefore$ $f(3,0)=9$ max value

and $\left.\begin{matrix} f(0,0)=0 \\ f(2,2)=0 \end{matrix}\right\}$ min value

(b) Find the greatest and smallest values of the function $f(x,y) = xy$ on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$.

Construct the Lagrange function:

$L(x,y,\lambda) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$

$\frac{\partial L}{\partial x} = y + \lambda\cdot\frac{2x}{8} = 0 \Rightarrow -y = \lambda\cdot\frac{x}{4} \Rightarrow \lambda = \frac{-4y}{x}$

$\frac{\partial L}{\partial y} = x + \lambda\cdot\frac{2y}{2} = 0 \Rightarrow -x = \lambda\cdot y \Rightarrow \lambda = \frac{-x}{y}$

$\Rightarrow \frac{-4y}{x} = \frac{-x}{y}$

$4y^2 = x^2$

$\frac{\partial L}{\partial \lambda} = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} - 1 = 0$

$y^2 = 1 \Rightarrow y = \mp 1$

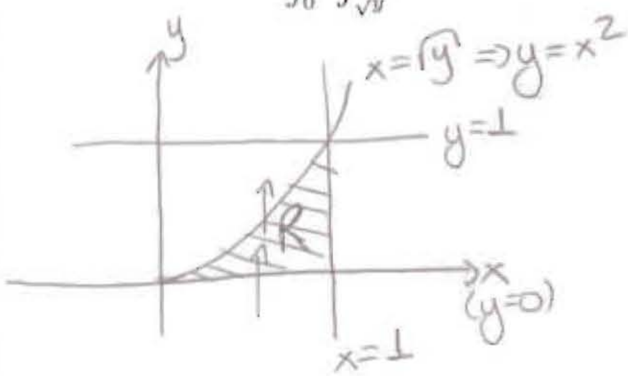
$x^2 = 4 \Rightarrow x = \mp 2$

$\left.\begin{matrix} f(2,1) = 2 \\ f(-2,-1) = 2 \end{matrix}\right\}$ max value

$\left.\begin{matrix} f(2,-1) = -2 \\ f(-2,1) = -2 \end{matrix}\right\}$ min value

2. (a) Sketch the domain of integration and evaluate the given integral:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3}\, dx dy.$$



$$= \int_0^1 \int_0^{x^2} \sqrt{1+x^3}\, dy\, dx$$

$$= \int_0^1 \sqrt{1+x^3} \left(y \Big|_0^{x^2} \right) dx$$

$$= \int_0^1 \sqrt{1+x^3} \cdot x^2\, dx$$

Let $u = 1+x^3 \Rightarrow du = 3x^2 dx \Rightarrow \frac{1}{3} du = x^2 dx$

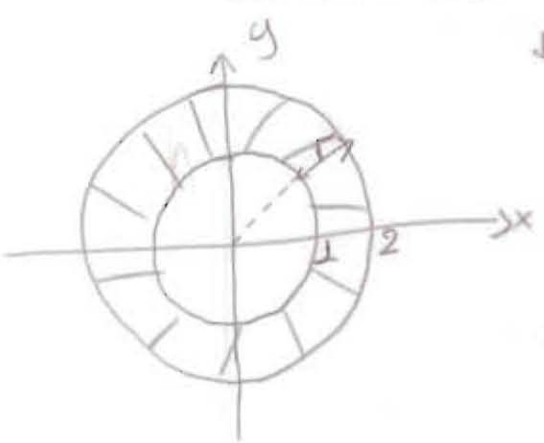
$x=0 \Rightarrow u=1$ and $x=1 \Rightarrow u=2$

$$= \int_1^2 \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} \Bigg|_1^2$$

$$= \frac{2}{9}\left(2^{3/2} - 1 \right)$$

(b) Evaluate the double integral:

$$\int_R \int \frac{2}{(1+x^2+y^2)^2}\, dA \text{ where } R \text{ is given by: } 1 \le x^2+y^2 \le 4.$$



Use polar coordinates!

$$\left. \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right\} \Rightarrow x^2+y^2 = r^2$$

$dA = r\,dr\,d\theta$

$1 \le x^2+y^2 \le 4$

$1 \le r^2 \le 4 \Rightarrow 1 \le r \le 2$ and $0 \le \theta \le 2\pi$

$$\int_1^2 \int_0^{2\pi} \frac{2}{(1+r^2)^2}\, r\, d\theta\, dr = 2\pi \int_1^2 \frac{2r}{(1+r^2)^2}\, dr$$

Let $u = 1+r^2$

$du = 2r\,dr$

$r=1 \Rightarrow u=2$

$r=2 \Rightarrow u=5$

$$= 2\pi \int_2^5 \frac{1}{u^2}\, du$$

$$= 2\pi \left(\frac{-1}{u} \right) \Bigg|_2^5 = -2\pi \left(\frac{1}{5} - \frac{1}{2} \right) = \frac{3\pi}{5}$$

3. (a) Evaluate the volume of the solid that lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$.

$r^2 = x^2+y^2 = 1$

$z = 4$ and $z = 1-x^2-y^2 = 1-r^2$.

Use the cylindrical coordinates:

$dV = r\,dr\,d\theta\,dz$

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 1-r^2 \le z \le 4 \end{cases}$$

The volume is:

$$V = \int_0^{2\pi} d\theta \int_0^1 dr \int_{1-r^2}^4 r\,dz = \int_0^{2\pi} d\theta \int_0^1 [\underbrace{4-(1-r^2)}_{3+r^2}]\, r\,dr = \int_0^{2\pi} d\theta \int_0^1 (3r+r^3)dr$$

$$= \int_0^{2\pi} \left(\frac{3r^2}{2} + \frac{r^4}{4} \Big|_0^1 \right) d\theta$$

$$= 2\pi \left[\frac{3}{2} + \frac{1}{4} \right]$$

$$= \frac{7\pi}{2}$$

(b) Convert the cartesian coordinates $(-\sqrt{3}, 3, 2)$ to

i. the cylindrical coordinates. $r^2 = x^2+y^2 = 3+9 = 12 \Rightarrow \boxed{r = 2\sqrt{3}}$

$\tan\theta = \frac{y}{x} = \frac{3}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$

$z = z = 2$

$\Rightarrow [r, \theta, z] = [2\sqrt{3}, \frac{2\pi}{3}, 2]$ ⊠

ii. the spherical coordinates.

$r^2 = x^2+y^2 \Rightarrow r = 2\sqrt{3}$

$\rho^2 = r^2+z^2 = 12+4 = 16 \Rightarrow \boxed{\rho = 4}$

$\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$

$\tan\phi = \frac{r}{z} = \frac{2\sqrt{3}}{2} = \sqrt{3} \Rightarrow \boxed{\phi = \frac{\pi}{3}}$

$\Rightarrow [\rho, \phi, \theta] = [4, \frac{\pi}{3}, \frac{2\pi}{3}]$ ⊠

4. (a) Solve the given differential equation: Let $\frac{y}{x} = \vartheta$ and

$$\frac{dy}{dx} = \frac{xy}{x^2 + 2y^2} \qquad \frac{dy}{dx} = \vartheta + x\frac{d\vartheta}{dx}.$$

$$\Rightarrow \vartheta + x\frac{d\vartheta}{dx} = \frac{x^2\left(\frac{y}{x}\right)}{x^2\left(1+2\frac{y^2}{x^2}\right)} = \frac{\vartheta}{1+2\vartheta^2}$$

So; $\vartheta + x\frac{d\vartheta}{dx} = \frac{\vartheta}{1+2\vartheta^2} \Rightarrow x\frac{d\vartheta}{dx} = \frac{\vartheta}{1+2\vartheta^2} - \vartheta$

$$\Rightarrow x\frac{d\vartheta}{dx} = \frac{\vartheta - \vartheta - 2\vartheta^3}{1+2\vartheta^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{1+2\vartheta^2}{-2\vartheta^3}d\vartheta \Rightarrow \int\frac{dx}{x} = -\int\frac{1+2\vartheta^2}{2\vartheta^3}d\vartheta$$

$$\Rightarrow \ln|x| = -\left[\int\frac{d\vartheta}{2\vartheta^3} + \int\frac{d\vartheta}{\vartheta}\right] = -\frac{1}{2}\frac{\vartheta^{-2}}{(-2)} - \ln|\vartheta| + C$$

$$\Rightarrow \ln|x| = \frac{1}{4\vartheta^2} - \ln|\vartheta| + C \Rightarrow \ln|x| = \frac{x^2}{4y^2} - \ln\left|\frac{y}{x}\right| + C \boxtimes$$

(b) Show that the given differential equation is exact, and solve it:

$$\underbrace{(ye^{xy} + 4y^3)}_{M}dx + \underbrace{(xe^{xy} + 12xy^2 - 2y)}_{N}dy = 0$$

Is $M_y = N_x$ (?)

$M_y = e^{xy} + yxe^{xy} + 12y^2$
$N_x = e^{xy} + xye^{xy} + 12y^2$] exact.

So if $\phi(x,y)$ is the solution; then $\phi_x = M$.

$$\Rightarrow \phi = \int(ye^{xy} + 4y^3)dx$$

$$\phi = y\frac{e^{xy}}{y} + 4y^3x + C_1(y) \Rightarrow \phi = e^{xy} + 4y^3x + C_1(y).$$

Since $\phi_y = N \Rightarrow xe^{xy} + 12y^2x + C_1'(y) = xe^{xy} + 12xy^2 - 2y$

$$\Rightarrow C_1'(y) = -2y \Rightarrow C_1(y) = -y^2.$$

$$\Rightarrow \phi(x,y) = e^{xy} + 4y^3x - y^2 + C \boxtimes$$