MATH 154
FINAL

STUDENT NOTES

GEÇMİŞ YILLARDA
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KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 06.06.2014

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

Name: ....

Student No: ....

Department: ....

Section: **Check for your instructor below:**

- [ ] Tahsin Öner
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1. (a) Find an equation of the tangent plane to the surface $w = f(x, y, z) = xyz + 2\ln(3x + y + z)$ at $(-1, 3, 1, f(-1, 3, 1))$.

$f(x,y,z) = xyz + 2\ln(3x+y+z)$ $\Rightarrow f(-1,3,1) = -3$

$f_x(x,y,z) = yz + 2 \cdot \frac{3}{3x+y+z}$ $\Rightarrow f_x(-1,3,1) = 9$

$f_y(x,y,z) = xz + 2 \cdot \frac{1}{3x+y+z}$ $\Rightarrow f_y(-1,3,1) = 1$

$f_z(x,y,z) = xy + 2 \cdot \frac{1}{3x+y+z}$ $\Rightarrow f_z(-1,3,1) = -1$

Equation of tangent plane:

$w = f(-1,3,1) + f_x(-1,3,1)(x-(-1)) + f_y(-1,3,1)(y-3) + f_z(-1,3,1)(z-1)$

$w = -3 + 9(x+1) + 1\cdot(y-3) - 1\cdot(z-1)$

(b) Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1.

Let $u = a\vec{i} + b\vec{j}$ be a unit direction vector $|u| = \sqrt{a^2+b^2} = 1$

$f_x(x,y) = 2x + \cos(xy)\cdot y$ $\Rightarrow f_x(1,0) = 2$

$f_y(x,y) = \cos(xy)\cdot x$ $\Rightarrow f_y(1,0) = 1$

$\nabla f(1,0) = f_x(1,0)\,\vec{i} + f_y(1,0)\cdot\vec{j} \Rightarrow \nabla f(1,0) = 2\vec{i} + \vec{j}$

$D_u f(1,0) = u \cdot \nabla f(1,0) = (a\vec{i} + b\vec{j})\cdot(2\vec{i}+\vec{j}) = 2a+b$

$D_u f(1,0)$ is given as 1 so $2a+b=1$ and we also know that $a^2+b^2=1$. Thus,

$$\left.\begin{array}{l} 2a+b=1 \\ a^2+b^2=1 \end{array}\right\} \Rightarrow \begin{array}{l} a=0,\ b=1 \\ a=\frac{4}{5},\ b=-\frac{3}{5} \end{array}$$

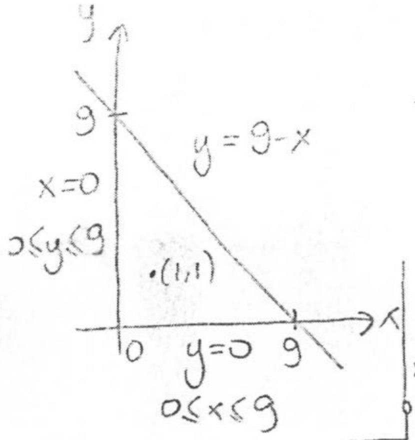
So, the direction vectors are

$u = \vec{j}$ and $u = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$

2. (a) Find the maximum and minimum values of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.

$$\left.\begin{array}{l} f_x(x,y) = 2 - 2x = 0 \Rightarrow x = 1 \\ f_y(x,y) = 2 - 2y = 0 \Rightarrow y = 1 \end{array}\right\} \Rightarrow$$

(1,1) is the critical point of $f(x,y)$, and $f(1,1) = 4$



Let's investigate the extreme values on the boundries:

On $y=9-x$; $0\le x\le 9$:

$f(x,9-x) = -2x^2+18x-61$

$x=0 \Rightarrow f(0,9) = -61$

$x=9 \Rightarrow f(9,0) = -61$

Critical point:

$(-2x^2+18x-61)' = 0$

$-4x+18 = 0 \Rightarrow x = 9/2$

$x=\frac{9}{2} \Rightarrow f\left(\frac{9}{2},\frac{9}{2}\right) = -\frac{41}{2}$

On $x=0$, $0\le y\le 9$:

$f(0,y) = 2+2y-y^2$

$y=0 \Rightarrow f(0,0) = 2$

$y=9 \Rightarrow f(0,9) = -61$

Critical point:

$(2+2y-y^2)' = 0$

$2-2y = 0 \Rightarrow y = 1$

$y=1 \Rightarrow f(0,1) = 3$

On $y=0$; $0\le x\le 9$:

$f(x,0) = 2+2x-x^2$

$x=0 \Rightarrow f(0,0) = 2$

$x=9 \Rightarrow f(9,0) = -61$

Critical point:

$(2+2x-x^2)' = 0$

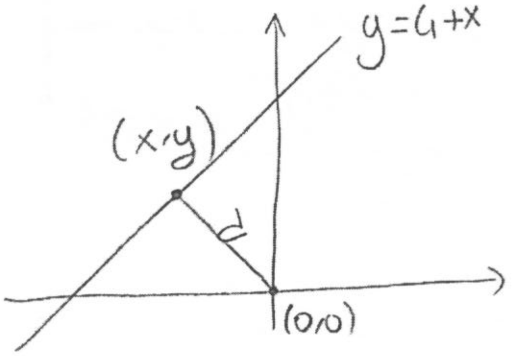
$2-2x = 0 \Rightarrow x = 1$

$x=1 \Rightarrow f(1,0) = 3$

Abs max : 4

Abs min : -61

(b) Find the shortest distance from the origin to the function $y = 4 + x$.



$d = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$

Minimize x^2+y^2
subject to $y-4-x=0$

$L(x,y,\lambda) = x^2+y^2+\lambda(y-4-x)$

$$\left.\begin{array}{l} \left.\begin{array}{l} \frac{\partial L}{\partial x} = 2x-\lambda = 0 \Rightarrow \lambda = 2x \\ \frac{\partial L}{\partial y} = 2y+\lambda = 0 \Rightarrow \lambda = -2y \end{array}\right\} \Rightarrow x = -y \\ \frac{\partial L}{\partial \lambda} = y-4-x = 0 \Rightarrow y-x = 4 \end{array}\right\} \Rightarrow y-(-y) = 4 \Rightarrow y = 2$$

$\Downarrow$

$x = -2$

$(x,y) = (-2,2)$

So, $d = \sqrt{(-2)^2+(2)^2} = \sqrt{8}$

3. (a) Evaluate the triple integral using the cylindrical coordinates $\iiint_R dV$ where R is the region given by $1 \leq z \leq \sqrt{4-(x^2+y^2)}$.

$x = r\cos\theta$ $\quad x^2+y^2 = r^2$ $\quad 1 \leq z \leq \sqrt{4-(x^2+y^2)} \Rightarrow 1 \leq z \leq \sqrt{4-r^2}$

$y = r\sin\theta$ $\quad dV = r\,dr\,d\theta\,dz$ $\quad 1 = \sqrt{4-(x^2+y^2)} \Rightarrow x^2+y^2 = 3 \Rightarrow 0 \leq r \leq \sqrt{3}$

$z = z$ $\quad 0 \leq \theta \leq 2\pi$

$$\iiint_R dV = \int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}}\int_0^{2\pi} r\,d\theta\,dz\,dr = \int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\left(\theta\Big|_0^{2\pi}\right)dz\,dr = 2\pi\int_0^{\sqrt{3}} r\cdot\underbrace{\left(z\Big|_1^{\sqrt{4-r^2}}\right)}_{=\sqrt{4-r^2}-1}dr$$

$$= 2\pi\underbrace{\int_0^{\sqrt{3}} r\cdot\sqrt{4-r^2}\,dr}\; - 2\pi\int_0^{\sqrt{3}} r\,dr = \cancel{2}\pi\int_4^1 \sqrt{u}\cdot\left(-\frac{1}{\cancel{2}}\right)du - 2\pi\int_0^{\sqrt{3}} r\,dr$$

Let $u = 4-r^2$

$du = -2r\,dr \Rightarrow -\frac{1}{2}du = r\,dr$

$r=0 \Rightarrow u=4$

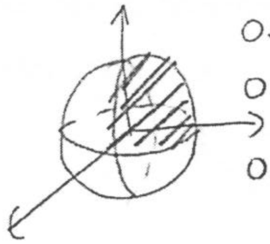
$r=\sqrt{3} \Rightarrow u=1$

$$= -\pi\cdot\left(\frac{u^{3/2}}{3/2}\Big|_4^1\right) - \cancel{2}\pi\cdot\frac{r^2}{\cancel{2}}\Big|_0^{\sqrt{3}}$$

$$= \frac{14\pi}{3} - 3\pi$$

$$= \frac{5\pi}{3}$$

(b) Evaluate the triple integral using the spherical coordinates $\iiint_B z\,dV$ where B is the region given by $x^2+y^2+z^2 \leq 9$, $x \geq 0$, $y \geq 0$, $z \geq 0$, (in the first octant).



$x = \rho\sin\phi\cos\theta$ $\quad x^2+y^2+z^2 = \rho^2$ $\quad 0 \leq \rho \leq 3$

$y = \rho\sin\phi\sin\theta$ $\quad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ $\quad 0 \leq \phi \leq \frac{\pi}{2}$

$z = \rho\cos\phi$ $\quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\iiint_B z\,dV = \int_0^{\pi/2}\int_0^3\int_0^{\pi/2} \rho\cos\phi\cdot\rho^2\sin\phi\,d\theta\,d\rho\,d\phi$$

$$= \int_0^{\pi/2}\int_0^3 \rho^3\cdot\cos\phi\cdot\sin\phi\cdot\underbrace{\left(\theta\Big|_0^{\pi/2}\right)}_{=\pi/2}d\rho\,d\phi$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\cos\phi\cdot\sin\phi\underbrace{\left(\frac{\rho^4}{4}\Big|_0^3\right)}_{=\frac{81}{4}}d\phi$$

$$= \frac{81\pi}{8}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi$$

$$= \frac{81\pi}{8}\int_0^{\pi/2}\frac{\sin(2\phi)}{2}d\phi$$

$$= \frac{81\pi}{16}\underbrace{\left(-\frac{\cos(2\phi)}{2}\Big|_0^{\pi/2}\right)}_{=1}$$

$$= \frac{81\pi}{16}$$

4. (a) Solve the given separable differential equation

$$\frac{dy}{dx} = \frac{x\,e^x}{y\sqrt{1+y^2}}$$

$y\sqrt{1+y^2}\,dy = x e^x\,dx$

$\int y\sqrt{1+y^2}\,dy = \int x e^x\,dx$

$t = 1+y^2$
$dt = 2y\,dy$
$\frac{1}{2}dt = y\,dy$
(substitution)

$u = x \quad dv = e^x dx$
$du = dx \quad v = e^x$
(Int. by parts; $\int u\,dv = uv - \int v\,du$)

$\Rightarrow \int \sqrt{t}\cdot\frac{1}{2}\,dt = xe^x - \int e^x dx$

$\frac{1}{2}\cdot\frac{t^{3/2}}{\frac{3}{2}} = xe^x - e^x + C$

$\frac{(1+y^2)^{3/2}}{3} = xe^x - e^x + C$

(b) Find the solution of the linear differential equation

$$\frac{dy}{dx} - 2xy = 3x^2 e^{x^2}$$

which satisfies the condition $y(0) = 5$.

$p(x) = -2x$, $q(x) = 3x^2 e^{x^2}$, $\mu(x) = \int p(x)\,dx = \int -2x\,dx = -x^2$

Multiply the equation by $e^{\mu(x)} = e^{-x^2}$:

$e^{-x^2}\frac{dy}{dx} + e^{-x^2}\cdot(-2x)\cdot y = e^{-x^2}\cdot 3x^2\cdot e^{x^2}$

$\frac{d}{dx}\left(e^{-x^2}\cdot y\right) = 3x^2$

$\int \frac{d}{dx}\left(e^{-x^2} y\right)dx = \int 3x^2\,dx$

$e^{-x^2} y = \frac{3x^3}{3} + C$

$y = e^{x^2}(x^3 + C)$

Since $y(0) = 5$, we have

$5 = e^0\cdot(0+C) \Rightarrow C = 5$

$\therefore y = y(x) = e^{x^2}(x^3+5)$

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MATH 154 CALCULUS II 30.05.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

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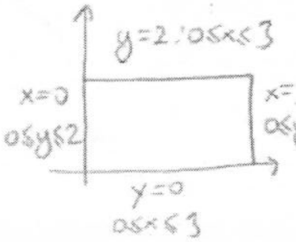
- [ ] Burak Ordin
- [ ] Güvenç Arslan
- [ ] Sinan Kapçak

1. (a) Find the maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle: $0 \le x \le 3$, $0 \le y \le 2$.

Find the critical points: $f_x(x,y)=0$, $f_y(x,y)=0$

$f_x(x,y) = 2x-2y=0 \Rightarrow x=y$; $f_y(x,y) = -2x+2 = 0 \Rightarrow x=1 \Rightarrow y=1$

$\therefore (1,1)$ is the only critical point, $f(1,1) = 1$



$x=0;\ 0\le y \le 2$:

$f(0,y) = 2y$

$f(0,0) = 0$

$f(0,2) = 4$

$f_y(0,y) = 2 \neq 0$

(no critical point)

$x=3;\ 0 \le y \le 2$:

$f(3,y) = 9-4y$

$f(3,0) = 9$

$f(3,2) = 1$

$f_y(3,y) = -4 \neq 0$ (no critical point)

$y=0;\ 0 \le x \le 3$

$f(x,0) = x^2$

$f_x(x,0) = 2x = 0 \Rightarrow x=0$

$f(0,0) = 0$, $f(3,0) = 9$

$y=2;\ 0 \le x \le 3$

$f(x,2) = x^2-4x+4$

$f_x(x,2) = 2x-4=0 \Rightarrow x=2$

$f(0,2) = 4$, $f(3,2) = 5$

$f(2,2) = 0$

$\therefore f(3,0) = 9$ max value

and $\left.\begin{array}{l} f(0,0)=0 \\ f(2,2)=0 \end{array}\right\}$ min value

(b) Find the greatest and smallest values of the function $f(x,y) = xy$ on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$.

Construct the Lagrange function:

$L(x,y,\lambda) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$

$\frac{\partial L}{\partial x} = y + \lambda \cdot \frac{2x}{8} = 0 \Rightarrow -y = \lambda \cdot \frac{x}{4} \Rightarrow \lambda = \frac{-4y}{x}$

$\frac{\partial L}{\partial y} = x + \lambda \cdot \frac{2y}{2} = 0 \Rightarrow -x = \lambda \cdot y \Rightarrow \lambda = \frac{-x}{y}$

$\Rightarrow \frac{-4y}{x} = \frac{-x}{y}$

$4y^2 = x^2$

$\frac{\partial L}{\partial \lambda} = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} - 1 = 0$

$y^2 = 1 \Rightarrow y = \mp 1$

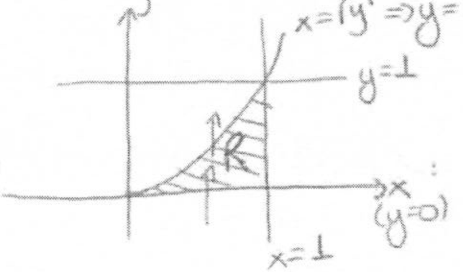
$x^2 = 4 \Rightarrow x = \mp 2$

$\left.\begin{array}{l} f(2,1) = 2 \\ f(-2,-1) = 2 \end{array}\right\}$ max value

$\left.\begin{array}{l} f(2,-1) = -2 \\ f(-2,1) = -2 \end{array}\right\}$ min value

2. (a) Sketch the domain of integration and evaluate the given integral:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3}\, dx dy.$$



$$= \int_0^1 \int_0^{x^2} \sqrt{1+x^3}\, dy\, dx$$

$$= \int_0^1 \sqrt{1+x^3} \left(y \Big|_0^{x^2} \right) dx$$

$$= \int_0^1 \sqrt{1+x^3} \cdot x^2\, dx$$

Let $u = 1+x^3 \Rightarrow du = 3x^2 dx \Rightarrow \frac{1}{3} du = x^2 dx$

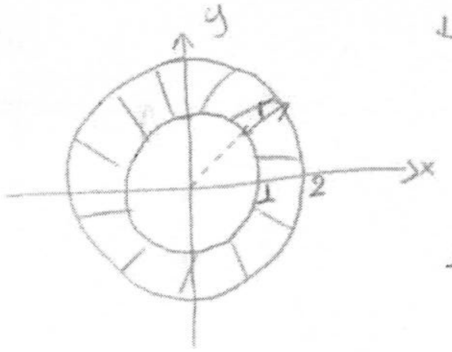
$x = 0 \Rightarrow u = 1$ and $x = 1 \Rightarrow u = 2$

$$= \int_1^2 \sqrt{u} \cdot \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} \Bigg|_1^2$$

$$= \frac{2}{9}\left(2^{3/2} - 1\right)$$

(b) Evaluate the double integral:

$$\int\int_R \frac{2}{(1+x^2+y^2)^2}\, dA \text{ where } R \text{ is given by: } 1 \le x^2+y^2 \le 4.$$



Use polar coordinates'

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow x^2+y^2 = r^2$

$dA = r\,dr\,d\theta$

$1 \le x^2+y^2 \le 4$

$1 \le r^2 \le 4 \Rightarrow 1 \le r \le 2$ and $0 \le \theta \le 2\pi$

$$\int_1^2 \int_0^{2\pi} \frac{2}{(1+r^2)^2}\, r\, d\theta dr = 2\pi \int_1^2 \frac{2r}{(1+r^2)^2} dr$$

Let $u = 1+r^2$

$du = 2r\,dr$

$r = 1 \Rightarrow u = 2$

$r = 2 \Rightarrow u = 5$

$$= 2\pi \int_2^5 \frac{1}{u^2} du$$

$$= 2\pi \left(\frac{-1}{u}\right)\Bigg|_2^5 = -2\pi\left(\frac{1}{5} - \frac{1}{2}\right) = \frac{3\pi}{5}$$

3. (a) Evaluate the volume of the solid that lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$.

$r^2 = x^2+y^2 = 1$

$z = 4$ and $z = 1-x^2-y^2 = 1-r^2$.

$dV = r\,dr\,d\theta\,dz$

Use the cylindrical coordinates: $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 1-r^2 \le z \le 4 \end{cases}$

The volume is :

$$V = \int_0^{2\pi} d\theta \int_0^1 dr \int_{1-r^2}^{4} r\,dz = \int_0^{2\pi} d\theta \int_0^1 [\underbrace{4-(1-r^2)}_{3+r^2}]\, r\,dr = \int_0^{2\pi} d\theta \int_0^1 (3r+r^3)\,dr$$

$$= \int_0^{2\pi} \left(\frac{3r^2}{2} + \frac{r^4}{4} \Big|_0^1 \right) d\theta$$

$$= 2\pi \left[\frac{3}{2} + \frac{1}{4} \right]$$

$$= \frac{7\pi}{2}$$

(b) Convert the cartesian coordinates $(-\sqrt{3}, 3, 2)$ to

i. the cylindrical coordinates. $r^2 = x^2+y^2 = 3+9 = 12 \Rightarrow \boxed{r = 2\sqrt{3}}$

$\tan\theta = \frac{y}{x} = \frac{3}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$

$z = z = 2$

$\Rightarrow [r, \theta, z] = [2\sqrt{3}, \frac{2\pi}{3}, 2]$ ⊠

ii. the spherical coordinates.

$r^2 = x^2+y^2 \Rightarrow r = 2\sqrt{3}$

$\rho^2 = r^2 + z^2 = 12 + 4 = 16 \Rightarrow \boxed{\rho = 4}$

$\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \frac{2\pi}{3}}$

$\tan\phi = \frac{r}{z} = \frac{2\sqrt{3}}{2} = \sqrt{3} \Rightarrow \boxed{\phi = \frac{\pi}{3}}$

$\Rightarrow [\rho, \phi, \theta] = [4, \frac{\pi}{3}, \frac{2\pi}{3}]$ ⊠

4. (a) Solve the given differential equation:

$$\frac{dy}{dx} = \frac{xy}{x^2+2y^2}$$

Let $\frac{y}{x} = \vartheta$ and $\frac{dy}{dx} = \vartheta + x\frac{d\vartheta}{dx}$.

$\Rightarrow \vartheta + x\frac{d\vartheta}{dx} = \frac{x^2(\frac{y}{x})}{x^2(1+2\frac{y^2}{x^2})} = \frac{\vartheta}{1+2\vartheta^2}$

So; $\vartheta + x\frac{d\vartheta}{dx} = \frac{\vartheta}{1+2\vartheta^2} \Rightarrow x\frac{d\vartheta}{dx} = \frac{\vartheta}{1+2\vartheta^2} - \vartheta$

$\Rightarrow x\frac{d\vartheta}{dx} = \frac{\vartheta - \vartheta - 2\vartheta^3}{1+2\vartheta^2}$

$\Rightarrow \frac{dx}{x} = \frac{1+2\vartheta^2}{-2\vartheta^3}d\vartheta \Rightarrow \int\frac{dx}{x} = -\int\frac{1+2\vartheta^2}{2\vartheta^3}d\vartheta$

$\Rightarrow \ln|x| = -\left[\int\frac{d\vartheta}{2\vartheta^3} + \int\frac{d\vartheta}{\vartheta}\right] = -\frac{1}{2}\frac{\vartheta^{-2}}{(-2)} - \ln|\vartheta| + C$

$\Rightarrow \ln|x| = \frac{1}{4\vartheta^2} - \ln|\vartheta| + C \Rightarrow \ln|x| = \frac{x^2}{4y^2} - \ln\left|\frac{y}{x}\right| + C$ ⊠

(b) Show that the given differential equation is exact, and solve it:

$$\underbrace{(ye^{xy}+4y^3)}_{M}dx + \underbrace{(xe^{xy}+12xy^2-2y)}_{N}dy = 0$$

Is $M_y = N_x$ (?)

$M_y = e^{xy} + yxe^{xy} + 12y^2$
$N_x = e^{xy} + xye^{xy} + 12y^2$] exact.

So if $\phi(x,y)$ is the solution; then $\phi_x = M$.

$\Rightarrow \phi = \int(ye^{xy}+4y^3)dx$

$\phi = y\frac{e^{xy}}{y} + 4y^3x + C_1(y) \Rightarrow \phi = e^{xy} + 4y^3x + C_1(y)$.

Since $\phi_y = N \Rightarrow xe^{xy} + 12y^2x + C_1'(y) = xe^{xy} + 12xy^2 - 2y$

$\Rightarrow C_1'(y) = -2y \Rightarrow C_1(y) = -y^2$.

$\Rightarrow \phi(x,y) = e^{xy} + 4y^3x - y^2 + C$ ⊠

2011
final exam

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MATH 154 CALCULUS II 03.06.2011

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

Name: .... #KEY# ....

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- ☐ Tahsin Öner
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- ☐ Sevin Gümgüm
- ☐ Ebru Özbilge

1. (a) Evaluate the given double integral over the quarter-disk Q given by $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 4$.

$$\iint_Q \frac{xy}{x^2+y^2}\,dA.$$ By using Polar coordinates:

$$\iint_Q \frac{xy}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^2 \frac{r\cos\theta \cdot r\sin\theta}{r^2}\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^2 r\sin 2\theta\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{r^2}{2}\Big|_0^2\right)\sin 2\theta\,d\theta = \int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= -\frac{1}{2}\cos 2\theta\,\Big|_0^{\pi/2} = -\frac{1}{2}\left[\cos\pi - \cos 0\right] = 1$$

(b) Evaluate the triple integral $\iiint_R xy^2 e^{-xyz}\,dV$ over the cube $0 \leq x, y, z \leq 1$.

$$\iiint_R xy^2 e^{-xyz}\,dV = \int_0^1\int_0^1\int_0^1 xy^2 e^{-xyz}\,dz\,dx\,dy$$

$$= \int_0^1\int_0^1 xy^2 \frac{e^{-xyz}}{-xy}\Big|_0^1\,dx\,dy = \int_0^1\int_0^1 y\left(1-e^{-xy}\right)dx\,dy$$

$$= \int_0^1\int_0^1\left(y - ye^{-xy}\right)dx\,dy = \int_0^1\left(xy - y\frac{e^{-xy}}{-y}\right)\Big|_0^1\,dy$$

$$= \int_0^1\left(xy + e^{-xy}\right)\Big|_0^1\,dy = \int_0^1\left(y + e^{-y} - 1\right)dy$$

$$= \left(\frac{y^2}{2} - e^{-y} - y\right)\Big|_0^1 = \frac{e-1}{2e}$$

2. (a) Find the volume of the region inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 20$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad z = x^2+y^2 \Rightarrow z = r^2 \qquad x^2+y^2+z^2 = 20 \Rightarrow r^2 + r^4 = 20$$

$$\Rightarrow r^4 + r^2 - 20 = 0 \Rightarrow (r^2+5)(r^2-4) = 0$$

$$\Rightarrow r = 2$$

$$V = \iiint dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{\sqrt{20-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\cdot(\sqrt{20-r^2} - r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r\sqrt{20-r^2} - r^3)\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(20-r^2)^{3/2} - \frac{r^4}{4}\right]\Bigg|_0^2 d\theta$$

$$= \int_0^{2\pi} \left[\left(-\frac{1}{3}\cdot 64 - 4\right) - \left(-\frac{1}{3}\cdot 20^{3/2}\right)\right] d\theta = \int_0^{2\pi} \left[-\frac{76}{3} + \frac{1}{3}\cdot 20^{3/2}\right] d\theta$$

$$= 2\pi\cdot\left(-\frac{76}{3} + \frac{1}{3}\cdot 20^{3/2}\right)$$

(b) Find $\iiint_B x^2+y^2+z^2\,dV$ where B is the upper-half of the ball $x^2+y^2+z^2 = 1$.

Spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$$x^2+y^2+z^2 = \rho^2$$

$$\iiint_B (x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin\phi\,d\phi\,d\theta = \frac{1}{5}\int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi/2} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \left[-\cos\frac{\pi}{2} + \cos 0\right] d\theta = \frac{1}{5}\int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{5}$$

3. (a) Verify that $y = cos3x$ and $y = sin3x$ are the solutions of the differential equation $y'' + 9y = 0$. Are any of the following functions solutions? (i) $y = 3sin3x - 9cos3x$, and (ii) $y = sin6x$.

$y=\cos(3x) \Rightarrow y'=-3\sin(3x)$ and $y''=-9\cos(3x)$

$y''+9y=-9\cos(3x)+9\cdot\cos(3x)=0$ Thus, $y=\cos(3x)$ is a sln.

$y=\sin(3x) \Rightarrow y'=3\cos(3x)$ and $y''=-9\sin(3x)$

$y''+9y=-9\sin(3x)+9\cdot\sin(3x)=0$ thus, $y=\sin(3x)$ is a sln.

$y=3\sin(3x)-9\cos(3x) \Rightarrow y'=9\cos(3x)+27\sin(3x)$

$y''=-27\sin(3x)+81\cos(3x)$

$y''+9y=-27\sin(3x)+81\cos(3x)+9(3\sin(3x)-9\cos(3x))=0$

$y=\sin(6x) \Rightarrow y'=6\cos(6x)$ and $y''=-36\sin(6x)$

$y''+9y=-36\sin(6x)+9\sin(6x)\neq 0$ Thus $y=\sin(6x)$ is not a solution.

(b) Solve the initial-value problem

$$\begin{cases} y' + (sinx)y = 2xe^{cosx} \\ y(\frac{\pi}{2}) = 1. \end{cases}$$

$p(x)=\sin x$, $q(x)=2xe^{\cos x}$

$\mu(x)=\int p(x)dx=\int \sin x\,dx=-\cos x$ and $e^{\mu(x)}=e^{-\cos x}$

Since the DE is a first-order linear nonhomogeneous, the solution is

$y=e^{-\mu(x)}\int e^{\mu(x)}\cdot q(x)dx$

$y=e^{\cos x}\int e^{-\cos x}\cdot 2xe^{\cos x}dx$

$y=e^{\cos x}\int 2x\,dx=e^{\cos x}(x+c)$

$y(\frac{\pi}{2})=1 \Rightarrow 1=\underbrace{e^{\cos\frac{\pi}{2}}}_{=1}(\frac{\pi}{2}+c) \Rightarrow c=1-\frac{\pi}{2}$

$\therefore y=e^{\cos x}(x+1-\frac{\pi}{2})$

4. (a) Solve the differential equation $x\frac{dy}{dx} = x\tan(\frac{y}{x}) + y$

$x \cdot \frac{dy}{dx} = x \cdot \tan\frac{y}{x} + y \Rightarrow \frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$ homogeneous eqn.

$v = \frac{y}{x} \Rightarrow y = v \cdot x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

$\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x} \Rightarrow \frac{dv}{dx} \cdot x + v = \tan v + v \Rightarrow \frac{dv}{dx} \cdot x = \tan v$

$\Rightarrow \cot v \cdot dv = \frac{dx}{x} \Rightarrow \ln|\sin v| = \ln|x| + \ln C \Rightarrow \ln|\sin v| = \ln|Cx|$

$\Rightarrow \sin v = Cx \Rightarrow \sin\left(\frac{y}{x}\right) = Cx$ or $y = x \cdot \sin^{-1}(Cx)$

(b) Solve the differential equation $(\frac{1}{x^2} + \frac{1}{y^2})dx + (\frac{-2x+1}{y^3})dy = 0$.

$M(x,y)dx + N(x,y)dy = 0 \quad \frac{\partial M}{\partial y} = -\frac{2}{y^3} = \frac{\partial N}{\partial x}$ exact.

we want to find $\phi(x,y) = C$ such that $\frac{\partial \phi}{\partial x} = M$

and $\frac{\partial \phi}{\partial y} = N$

$\frac{\partial \phi}{\partial x} = M \Rightarrow \phi(x,y) = \int M(x,y)dx + C_1(y)$

$= \int\left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + C_1(y) = -\frac{1}{x} + \frac{x}{y^2} + C_1(y)$

$\frac{\partial \phi}{\partial y} = N \Rightarrow \cancel{-\frac{2x}{y^3}} + C_1'(y) = \cancel{-\frac{2x}{y^3}} + \frac{1}{y^3} \Rightarrow C_1(y) = \int \frac{1}{y^3} dy$

$\Rightarrow C_1(y) = -\frac{1}{2y^2} + C_2$

$\phi(x,y) = -\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_2 = C$

or

$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = K.$

2010
Final exam

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 09.06.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

Name: ....

Student No: ....

Department: ....

Section: **Check for your instructor below:**

☐ Murat Adıvar

☐ Sevin Gümgüm

☐ Tahsin Öner

☐ Sinan Kapçak

Duration: 110 mins

1. (a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 2y^2}$.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xy}{x^2+2y^2} \quad \text{Let } y = vx \\
v + x\frac{dv}{dx} &= \frac{vx^2}{(1+2v^2)x^2} \\
x\frac{dv}{dx} &= \frac{v}{1+2v^2} - v = -\frac{2v^3}{1+2v^2} \\
\int \frac{1+2v^2}{v^3}\,dv &= -2\int \frac{dx}{x} \\
-\frac{1}{2v^2} + 2\ln|v| &= -2\ln|x| + C_1 \\
-\frac{x^2}{2y^2} + 2\ln|y| &= C_1 \\
x^2 - 4y^2\ln|y| &= Cy^2.
\end{aligned}$$

(b) Solve the differential equation $(e^x \cos y + 2x)dx + (e^{-x}\sin y + 2y)dy = 0$.

2009
Final exam

1. (a) Starting with the power series representation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$$

determine the Taylor series representation of $f(x) = \dfrac{1}{x^2}$ in powers of $x - 5$.

Solution:

First, differentiate the function $\dfrac{1}{1-x}$ and its series representation with respect to x to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \textstyle\sum_{n=1}^{\infty} n\, x^{n-1}.$$

Let $t = x - 5$, i.e., $x = 5 + t$. So, we have

$$f(x) = \frac{1}{x^2} = \frac{1}{(5+t)^2} = \frac{1}{25\left(1 + \frac{t}{5}\right)^2}.$$

Substitute $\dfrac{-t}{5}$ for x in $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$, to get

$$\frac{1}{\left(1 + \dfrac{t}{5}\right)^2} = \textstyle\sum_{n=1}^{\infty} n \left(\dfrac{-t}{5}\right)^{n-1}.$$

Then multiply the resulting equation by $\dfrac{1}{25}$ to arrive at

$$\frac{1}{25\left(1 + \dfrac{t}{5}\right)^2} = \frac{1}{25} \textstyle\sum_{n=1}^{\infty} n \left(\dfrac{-t}{5}\right)^{n-1}.$$

Finally, use the transformation $t = x - 5$ to find

$$\frac{1}{x^2} = \textstyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}\, n}{5^{n+1}} (x-5)^{n-1}.$$

(b) If $S(x) = \int_0^x \sin(t^2)\, dt$, find $\lim_{x\to 0} \dfrac{x^3 - 3S(x)}{x^7}$

Solution:

Maclaurin series for $\sin(t^2) = t^2 - \dfrac{t^6}{3!} + \dfrac{t^{10}}{5!} - ...$

$$S(x) = \textstyle\int_0^x \sin(t^2)\, dt = \int_0^x \left(t^2 - \dfrac{t^6}{3!} + \dfrac{t^{10}}{5!} - ...\right) dt$$

$$S(x) = \left(\frac{t^3}{3} - \frac{t^7}{7\times 3!} + \frac{t^{11}}{11 \times 5!} - ...\right)\Big|_0^x$$

$$S(x) = \frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} - ...$$

Substitute $S(x)$ in the limit:

$$\lim_{x\to 0}\frac{x^3-3S(x)}{x^7}=\lim_{x\to 0}\frac{x^3-3\Big(\frac{x^3}{3}-\frac{x^7}{7\times 3!}+\frac{x^{11}}{11\times 5!}-...\Big)}{x^7}$$

$$=\lim_{x\to 0}\Big(\frac{3}{7\times 3!}-\frac{3\,x^4}{11\times 5!}+...\Big)=\frac{3}{7\times 3!}=\frac{1}{14}$$

2. Find the Fourier series of the function $f(t)$ with period 2 whose values in the interval $[-1,1)$ are given by

$$f(t)=\begin{cases}0 & \text{if } 1\le t<0\\ t & \text{if } 0\le t<1\end{cases}.$$

Solution:
The Fourier coefficients of f are as follows:

$$\frac{a_0}{2}=\frac{1}{2}\int_{-1}^{1}f(t)dt=\frac{1}{2}\int_0^1 tdt=\frac{1}{4},$$

$$\begin{aligned}a_n&=\int_{-1}^{1}f(t)\cos(n\pi t)dt\\&=\int_0^1 t\cos(n\pi t)dt\\&=\frac{(-1)^n-1}{n^2\pi^2}\\&=\begin{cases}-2/(n\pi)^2 & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases},\end{aligned}$$

and

$$b_n=\int_0^1 t\sin(n\pi t)dt=\frac{-(-1)^n}{n\pi}.$$

Hence, the Fourier series of f is

$$\frac{1}{4}-\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)\pi t)-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin(k\pi t).$$

3. (a) Find the general solution to $x\dfrac{dy}{dx}+3y=6x^3$.

Solution:
$\dfrac{dy}{dx}+\dfrac{3}{x}y=6x^2.$

$\mu(x)=e^{\int\frac{3}{x}dx}=x^3.$

Multiply both sides by $\mu(x)$ to obtain

$x^3\dfrac{dy}{dx}+3x^2y=6x^5.$

That is, $(x^3y)'=6x^5$. Integrate both sides to get $y=x^3+Cx^{-3}$.

4. (a) Use cylindrical coordinates to evaluate the volume of the region between paraboloids $z = 16 - x^2 - y^2$ and $z = x^2 + y^2 - 2$.

Solution

$$x = r\sin\theta, \quad y = r\cos\theta, \quad z = z$$

$$16 - x^2 - y^2 = x^2 + y^2 - 2, \quad x^2 + y^2 = 9, \quad r = 3$$

$$\iint_R (16 - x^2 - y^2 - (x^2 + y^2 - 2))dA = \int_0^{2\pi}\int_0^3 (18 - 2r^2) r\,dr\,d\theta$$

$$= \int_0^{2\pi} (9r^2 - \frac{r^4}{2})\Big|_0^3 d\theta = \int_0^{2\pi} \frac{81}{2} d\theta = 81\pi \text{ cubic units.}$$

(b) Find $\iiint_B (x^2 + y^2)\,dV$, where B is the ball given by $x^2 + y^2 + z^2 \le a^2$.

Solution

$$\iiint_B (x^2 + y^2)\,dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\,d\phi \int_0^a R^2 \sin^2\phi R^2\,dR$$

$$= 2\pi \int_0^{\pi} \sin^3\phi\,d\phi \int_0^a R^4\,dR = 2\pi\frac{a^5}{5}\int_0^{\pi} \sin^3\phi\,d\phi = 2\pi\frac{a^5}{5}\int_0^{\pi} (1-\cos^2\phi)\sin\phi\,d\phi$$

$$(\cos\theta = u, \quad \sin\theta d\theta = -du)$$

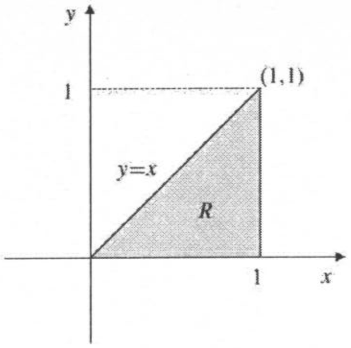
$$= 2\pi\frac{a^5}{5}\int_1^{-1} (u^2 - 1)\,du = 2\pi\frac{a^5}{5}(\frac{u^3}{3} - u)\Big|_1^{-1}$$

$$= 2\pi\left(\frac{4}{3}\right)\frac{a^5}{5} = \frac{8\pi a^5}{15}.$$

2. (a) Evaluate the iterated integral $\int_0^1 dy \int_y^1 e^{-x^2}\,dx$.

Solution

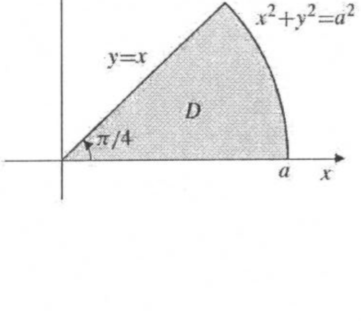
$$\int_0^1 dy \int_y^1 e^{-x^2} dx = \int_R e^{-x^2} dA \quad (R \text{ as shown})$$
$$= \int_0^1 e^{-x^2} dx \int_0^x dy$$
$$= \int_0^1 xe^{-x^2} dx \quad \text{Let } u = x^2, \quad du = 2xdx$$
$$= \frac{1}{2}\int_0^1 e^{-u} du = -\frac{1}{2}e^{-u}\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$



(b) Evaluate $\iint_D xy\,dA$, where D is the plane region satisfying $x \geq 0$, $0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

Solution

$$\iint_D xydA = \int_0^{\pi/4} d\theta \int_0^a r\cos\theta r\sin\theta r dr$$
$$= \frac{1}{2}\int_0^{\pi/4} \sin 2\theta d\theta \int_0^a r^3 dr$$
$$= \frac{a^4}{8}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{a^4}{16}.$$



3. (a) Find the area of the region in the first quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$ and $y = 2x$.

Solution

Let $u = xy$, $v = y/x$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = 2\frac{y}{x} = 2v,$$

so that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$. The region D in the first quadrant of the xy-plane bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$ corresponds to the rectangle R in the uv-plane bounded $u = 1$, $u = 4$, $v = 1$, and $v = 2$. Thus the area of D is given by

$$\iint_D dxdy = \iint_R \frac{1}{2v} dudv = \frac{1}{2}\int_1^4 du \int_1^2 \frac{dv}{v} = \frac{3}{2}\ln 2 \text{ sq. units.}$$

(b) Evaluate the triple integral $\iiint_R y\, dV$, where R is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{1-(x+y)} y\, dzdydx = \int_0^1 \int_0^{1-x} y(z|_0^{1-(x+y)})dydx$$

$$= \int_0^1 \int_0^{1-x} (y - xy - y^2)dydx = \int_0^1 (\frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3})|_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^3}{6} dx = \frac{-(1-x)^4}{24}|_0^1 = \frac{1}{24}$$

FİNAL ÖĞRENCİ NOTLAR

=LAGRANGE MULTIPLIERS=

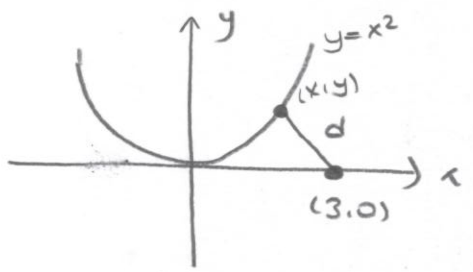
To find candidates for points on the curve $g(x,y)=0$ at which $f(x,y)$ is max or min, we should look for critical points of Lagrangian function

$$\boxed{L(x,y,\lambda) = f(x,y) + \lambda g(x,y)}$$

At any critical point of L we have,

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

Ex Find the shortest distance from the point $(3,0)$ to the parabola $y=x^2$ by using the method of Lagrange multipliers.



The distance from $(3,0)$ to (x,y)

$$d = \sqrt{(x-3)^2 + (y-0)^2}$$

$$\Leftrightarrow d^2 = (x-3)^2 + (y-0)^2$$

$$= f(x,y)$$

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$y = x^2 \iff \underbrace{y - x^2 = 0}$$

$$g(x,y) = y - x^2$$

$$L(x,y,\lambda) = (x-3)^2 + y^2 + \lambda \cdot (y - x^2)$$

For CP

$$L(x,y,\lambda) = (x-3)^2 + y^2 + \lambda (y - x^2)$$

$$\frac{\partial L}{\partial x} = 2(x-3) + \lambda(-2x) = 0$$

$$\Rightarrow x - 3 - \lambda x = 0$$

$$\Rightarrow \lambda = \frac{x-3}{x}$$

$$\frac{\partial L}{\partial y} = 2y + \lambda = 0$$

$$\Rightarrow \lambda = -2y$$

$$\frac{\partial L}{\partial \lambda} = y - x^2 = 0$$

$$\lambda = -2y = \frac{x-3}{x} \iff -2xy = x - 3$$

$$y = x^2$$

$$\Rightarrow -2x \cdot x^2 = x - 3$$

$$2x^3 + x - 3 = 0$$

$\Rightarrow$ derece 3,4,5 ise polinomda sabit sayının bölenlerine bakarız. Bölenlerden biri köktür. Diğer çarpanı bulmak için o köklere böleriz.

$2x^3+x-3=0$

Let $x=1$ $\quad 2\cdot1^3+1-3=0$

$\boxed{x=1}$ is the root

$$\begin{array}{l|l} 2x^3+x-3 & x-1 \\ \hline -\ 2x^3-2x^2 & 2x^2+2x+3 \\ \hline 2x^2+x-3 & \\ -\ 2x^2-2x & \\ \hline 3x-3 & \\ -\ 3x-3 & \\ \hline 0 & \end{array}$$

$2x^3+x-3=(x-1)\underbrace{(2x^2+2x+3)}$

$\Delta=b^2-4ac<0$

$\Downarrow$

no root

$\left.\begin{array}{l} x=1 \\ y=x^2 \end{array}\right\} \quad y=1^2=1 \quad (1,1)$

$d^2=f(1,1)=(x-3)^2+y^2$

$=(1-3)^2+1^2=5$

$\underline{\underline{d=\sqrt{3}}}$

EX (2009, MT). Find the distance from the origin to the plane $x+2y+2z=3$ by using the method of Lagrange multipliers.

———o———

Let (x,y,z) be the point on the plane. Then we write the distance from $(0,0,0)$ to (x,y,z)

$$d=\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$$=\sqrt{x^2+y^2+z^2}$$

$$d^2=f(x,y,z)=x^2+y^2+z^2$$

$$x+2y+2z=3 \Longleftrightarrow \underbrace{x+2y+2z-3=0}_{g(x,y,z)}$$

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda g(x,y,z)$$

$$=x^2+y^2+z^2+\lambda[x+2y+2z-3]$$

For CP

$$\frac{\partial L}{\partial x}=2x+\lambda=0$$

$$\Rightarrow x=\frac{-\lambda}{2}$$

OFiSSER Kırtasiye - Dijital Baskı
Not Merkezi

$$\frac{\partial L}{\partial y} = 2y + 2\lambda = 0$$

$$\Rightarrow y = -\lambda$$

$$\frac{\partial L}{\partial z} = 2z + 2\lambda = 0$$

$$\Rightarrow z = -\lambda$$

$$x + 2y + 2z - 3 = 0$$

$$-\frac{\lambda}{2} - 2\lambda - 2\lambda - 3 = 0$$

$$-9\lambda - 6 = 0$$

$$\lambda = \frac{-2}{3}$$

$$\frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 = 0$$

For $\lambda = -2/3$

$$x = \frac{-\lambda}{2} = \frac{-(-2/3)}{2} = \frac{1}{3}$$

$$y = z = -\lambda = -\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$$

$$d^2 = f\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1 \Rightarrow \boxed{d = 1}$$

EX (2012 - Final Exam)

Find the greatest and smallest values of the function $f(x,y) = xy$ on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$= xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$$

For CP

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial \lambda} = 0$$

$$L(x,y,\lambda) = xy + \lambda\left[\frac{x^2}{8} + \frac{y^2}{2} - 1\right]$$

$$\frac{\partial L}{\partial x} = y + \lambda \cdot \frac{y}{4} = 0$$

$$\Rightarrow 4y + \lambda x = 0$$

$$\Rightarrow \lambda = -\frac{4y}{x}$$

$$\frac{\partial L}{\partial y} = x + \lambda y = 0$$

$$\Rightarrow \lambda = \frac{-x}{y}$$

y

$$\frac{\partial L}{\partial \lambda} = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\Downarrow$$

$$\lambda = -\frac{4y}{x} = -\frac{x}{y} \Rightarrow 4y^2 = x^2 = m$$

$$\Rightarrow x^2 = m$$

$$\Rightarrow y^2 = m/4$$

$$\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{m}{8} + \frac{m/4}{2} = 1$$

$$\frac{m}{8} + \frac{m}{8} = 1 \qquad m = 4$$

$$4y^2 = x^2 = m = 4$$

$x^2 = 4$ $\qquad$ $4y^2 = 4$

$\Downarrow$ $\qquad$ $y^2 = 1$

$x = \mp 2$ $\qquad$ $y = \mp 1$

$f(x, y) = xy$ $\qquad$ $(1,2)$

$f(1,2) = 2$ $\qquad$ $(1,-2)$ } greatest

$f(-1,-2) = 2$ $\qquad$ $(-1,2)$

$f(-1,2) = -2$ $\qquad$ $(-1,-2)$ } smallest

$f(1,-2) = -2$

=DOUBLE INTEGRAL=

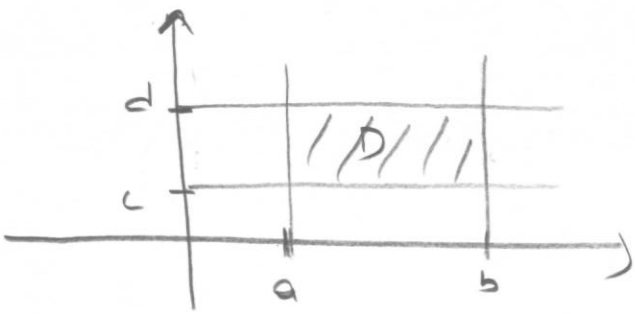
If $f(x,y)$ is defined and bounded on the domain D, we say that f is integrable over D and the double integral of f over D is given by

$$\iint_D f(x,y)\,dA$$

where $dA = dx\,dy$ or $dA = dy\,dx$

Property $\Rightarrow$

$$D = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$$



$$\iint_D f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

OR

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex Evaluate the double integral by iteration

$$\iint_D (x^2+y^2)\,dA$$

where D is the rectangle $0 \le x \le 1$, $0 \le y \le 2$

The first solution method

$$\iint_D (x^2+y^2)\,dA = \int_0^1 \underbrace{\int_0^2 (x^2+y^2)\,dy}\,dx$$

$$= \int_0^1 \left(x^2y^2 + \frac{y^3}{3}\right)\Big|_0^2 = \int_0^1 \left[x^2(2-0) + \frac{1}{3}(2^3-0^3)\right]dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3}\right)dx = \frac{2x^3}{3} + \frac{8x}{3}\Big|_0^1 = \frac{10}{3}$$

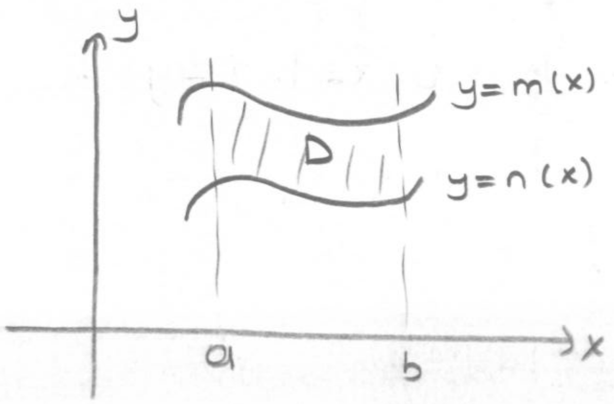
Second method;

$$\iint_D (x^2+y^2)\,dA = \int_0^2\int_0^1 (x^2+y^2)\,dx\,dy = \int_0^2 \left(\frac{x^3}{3} + xy^2\right)dy\Big|_0^1$$

$$= \int_0^2 \left(\frac{1}{3} + y^2\right)dy$$

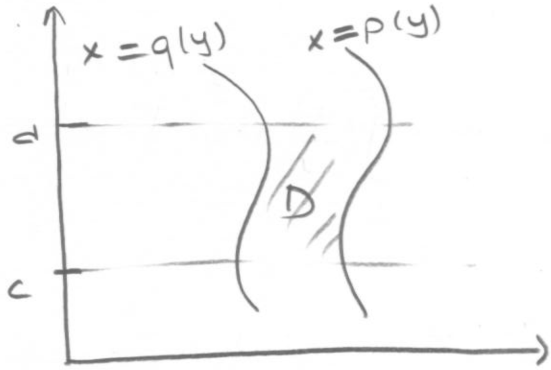
$$= \frac{y}{3} + \frac{y^3}{3}\Big|_0^2 = \frac{2}{3} + \frac{8}{3} = \frac{10}{3}$$

Iteration of Double Integrals in Cartesian coordinates



$$D = \begin{cases} a \le x \le b \\ n(x) \le y \le m(x) \end{cases}$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{n(x)}^{m(x)} f(x,y)\,dy\,dx = \int_a^b dx \int_{n(x)}^{m(x)} f(x,y)\,dy$$



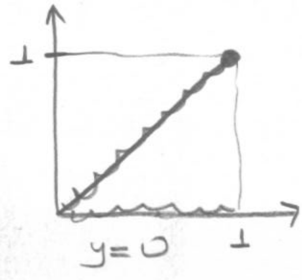
$$D = \begin{cases} c \le y \le d \\ q(y) \le x \le p(y) \end{cases}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{q(y)}^{p(y)} f(x,y)\,dx\,dy$$

$$= \int_c^d dy \int_{q(y)}^{p(y)} f(x,y)\,dx \quad //$$

Ex Evaluate $\iint_T xy\,dA$ over the triangle T with vertices $(0,0), (1,0), (1,1)$

First Method =)



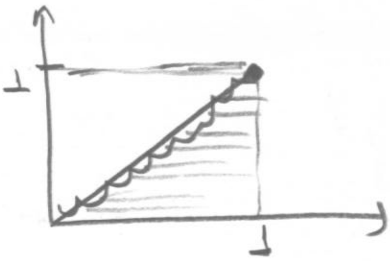
$$\underbrace{\iint_T f(x,y)\,dy\,dx}=$$

$$= \int_0^1 \int_0^x xy\,dy\,dx$$

$$= \int_0^1 \frac{xy^2}{y}\Big|_0^x\,dx$$

$$= \int_0^1 \frac{x^3}{2}\,dx = \frac{x^4}{8} = \frac{1}{8}$$

Second metHOD



$$\iint_T xy\,dx\,dy = \int_0^1 \underline{\int_y^1 xy\,dx}\,dy$$

$$= \int_0^1 \frac{xy^2}{2}\Big|_y^1 dy$$

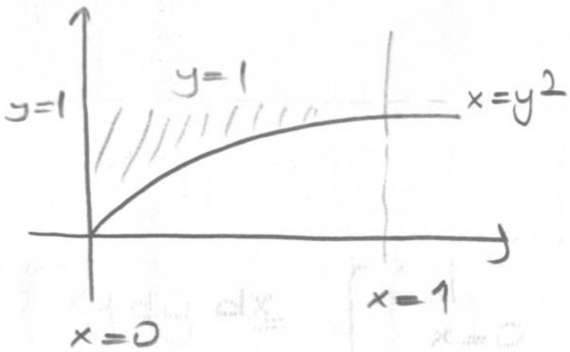
$$= \int_0^1 \frac{y}{2}\,(1-y^2)\,dy$$

$$= \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right) dy = \frac{1}{8} \,//$$

(EX) Evaluate the iterated integral

$$\int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3} dy = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dy\, dx$$

$$D: \begin{cases} 0 \leq x \leq 1 \\ \sqrt{x} \leq y \leq 1 \end{cases} \qquad \begin{aligned} y &= \sqrt{x} \\ y^2 &= x \end{aligned}$$



$$\rightarrow \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dx\,dy = \int_0^1 \int_0^{y^2} e^{y^3} dx\,dy$$

$$= \int_0^1 e^{y^3} x \Big|_0^{y^2} = \int_0^1 e^{y^3} (y^2 - 0)\, dy$$

$$= \int_0^1 y^2 e^{y^3} dy \qquad \begin{aligned} y^3 &= u \\ 3y^2\, dy &= du \end{aligned}$$

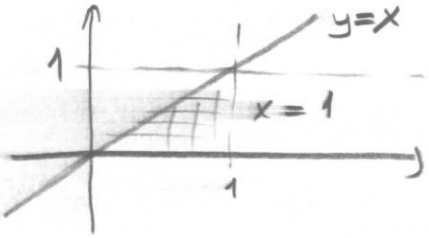
$$\frac{1}{3} \int_0^1 du\, e^u \qquad \frac{e^{y^3}}{3} \Big|_0^1 = \frac{e}{3} - \frac{1}{3}$$

(EX) Sketch the domain of integration and evaluate the given iterated integral

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$

$$R: \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$$

$\int e^{-x^2}\, dx \rightarrow$ not possible to evaluate



$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy = \int_0^1 \int_0^x e^{-x^2}\, dx\, dy$$

$-x^2 = u$

$-2x\, dx = du$

$$= \int_0^1 y \cdot e^{-x^2}\, dx \Big|_0^x$$

$$\int_0^1 e^{-x^2}\, dx = \frac{-1}{2} \int_0^1 du\, e^u$$

$$\frac{-e^{-x^2}}{2} \Big|_0^1$$

$$\frac{-e^{-x^2}}{2} \Big|_0^1 = -\frac{e}{2} + \frac{e^0}{2} = -\frac{e}{2} + \frac{1}{2} //$$

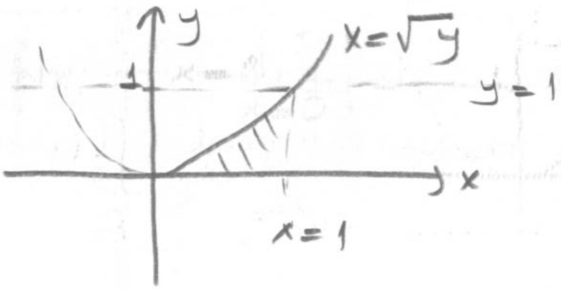
EX (2012, Final Exam)

Sketch the domain of integration and evaluate the given integral

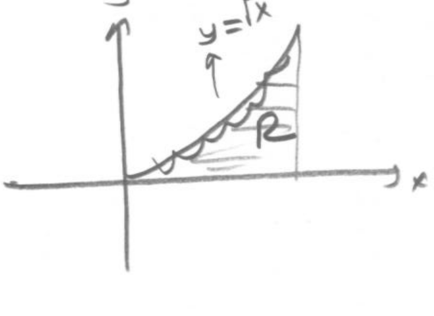
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3}\, dx\, dy$$

$$R: \begin{cases} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{cases}$$

$x = \sqrt{y} \iff x^2 = y$



$\int \sqrt{1+x^3}\, dx$ (not possible to evaluate this)



$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3}\, dx\, dy =$$

$$= \int_0^1 \int_0^{x^2} \sqrt{1+x^3}\, dy\, dx$$

$\sqrt{y} = x \iff y = x^2$

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$$= \int_0^1 \sqrt{1+x^3}\, y \Big|_0^{x^2} dx = \underbrace{\int_0^1 x^2 \sqrt{1+x^3}\, dx}$$

$$1 + x^2 = u$$

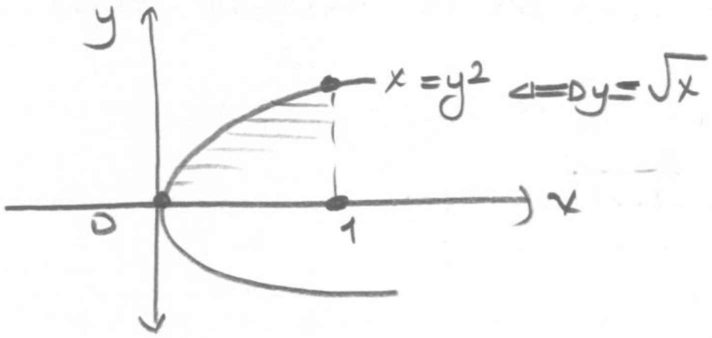
$$x^2\, dx = \frac{du}{3}$$

when $x=0$ $\quad u = 1+0^3 = 1$

" $\quad x=1$ $\quad u = 1+1^3 = 2$

$$= \int_1^2 \sqrt{u}\, \frac{du}{3} = \frac{1}{3} \left. \frac{u}{\frac{3}{2}} \right|_1^2 = \frac{2}{9}\left(2^{3/2} - 1\right)$$

EX (2011 midterm) Evaluate the double integral $\iint_R (x^2+y^2)\, dA$ where R is the finite region in the first quadrant bounded by the curves $y=x^2$ and $x=y^2$



$$\left.\begin{array}{l} y = x^2 \\ x = y^2 \end{array}\right\} \Rightarrow \begin{array}{l} x = y^2 = (x^2)^2 = x^4 \\ x - x^4 = 0 \\ x = 0 \quad x = 1 \end{array}$$

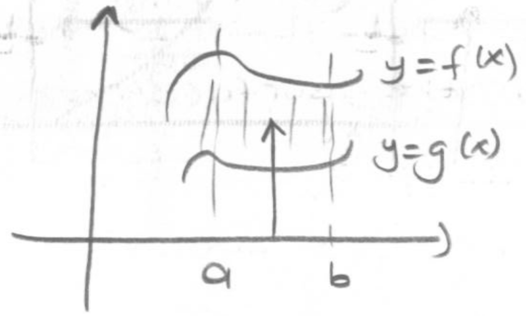
$dA = dx\,dy$ or $dA = dy\,dx$

$$\iint_R (x^2+y^2)\, dA = \int_0^1 \underbrace{\int_x^{\sqrt{x}} (x^2+y^2)\, dy}\, dx$$

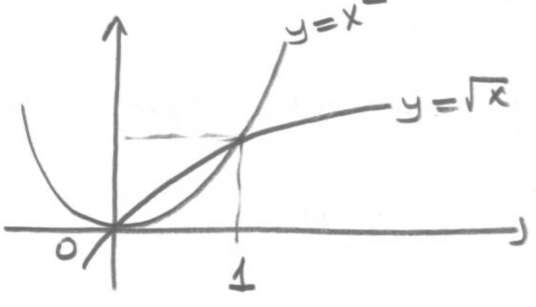
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Bigg|_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[x^2(\sqrt{x}-x^2) + \frac{1}{3}\left[(\sqrt{x})^3-(x^2)^3\right]\right] dx$$

$$= \frac{6}{35}$$



$$\int_a^b \int_{g(x)}^{f(x)} dy\, dx$$



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Properties of Double Integrals ⟹

1. (Area of a Domain)

$$\iint_D dA = \text{area of } D$$

2. (Integrals representing volume)

(a) If $f(x,y) \geq 0$ on D

then
$$\iint_D f(x,y)\,dA = V \geq 0$$

where V is the volume of the solid using vertically above D and below the surface

$$z = f(x,y)$$

(b) If $f(x,y) \leq 0$ on D, then

$$\iint_D f(x,y)\,dA = -V \leq 0$$

where V is the volume of the solid lying below D and above the surface $z = f(x,y)$

Ex Find the volume of the solid lying above the square Q defined by $0 \leq x \leq 1,\ 0 \leq y \leq 2$ and below the plane $z = 4 - x - y$

$$V = \iint_{Q} f(x,y)\,dA$$

$$z = f(x,y) = 4 - x - y$$

$$= \iint_{Q} (4 - x - y)\,dA = \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx$$

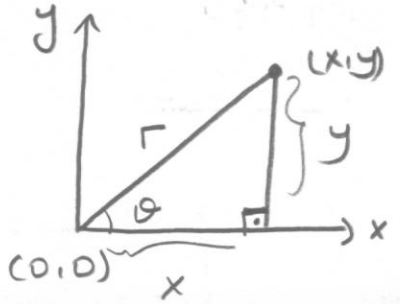
$$= \int_0^1 \left[4y - xy - \frac{y^2}{2} \right]\Big|_0^2 \,dx$$

$$= \int_0^1 [8 - 2x - 2]\,dx$$

$$= (6x - x^2)\Big|_0^1 = 6 \,//$$

Double Integrals in Polar Coordinates

Given a point (x,y) in cartesion coordinate

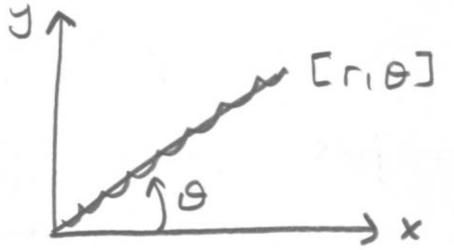


$$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$$

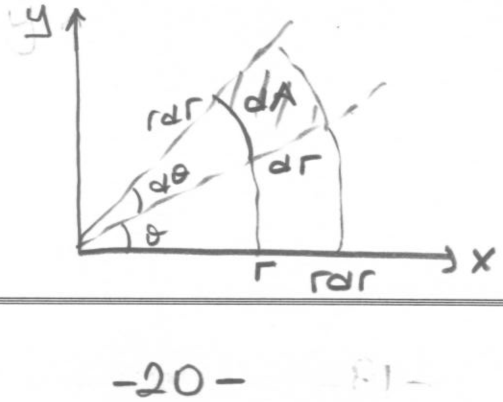
$$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$$

$$\boxed{\tan\theta = \frac{y}{x} \Leftrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

In this case we write the polar coordinates $[r,\theta]$

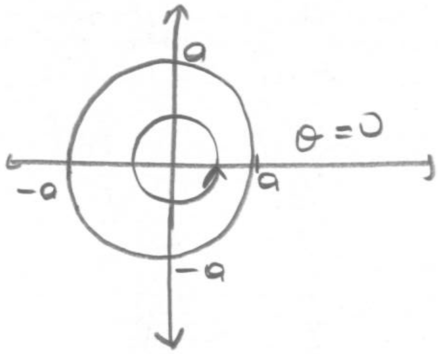


$dA = dxdy$ in cartesion coordinate $\boxed{dA = r\,dr\,d\theta}$ in polar coordinate



EX

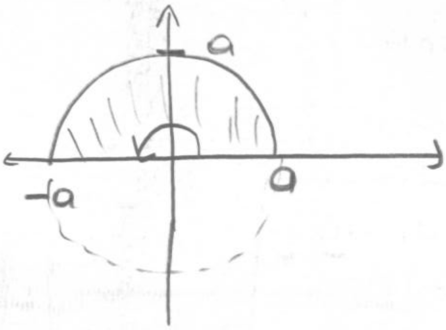
ⓐ



$x^2+y^2 \le a^2$, $0 \le r \le a$

$0 \le \theta \le 2\pi$

ⓑ

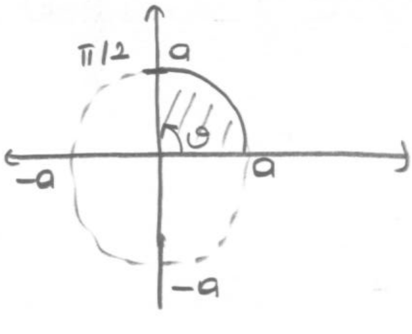


$0 \le r \le a$

$0 \le \theta \le \pi$

(semi circle)

ⓒ

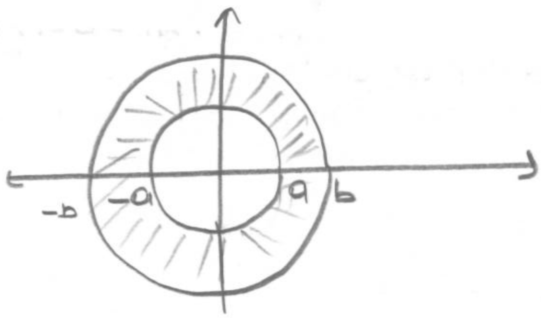


$x^2+y^2 \le a^2$, $0 \le r \le a$

$x \ge 0$ $\qquad$ $0 \le \theta \le \pi/2$

$y \ge 0$

ⓓ

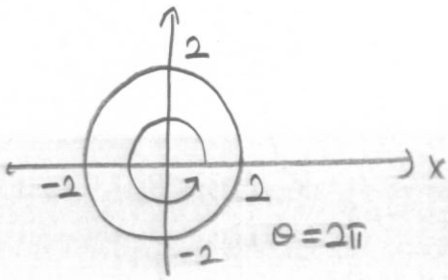


$a^2 \le x^2+y^2 \le b^2$

, $0 \le \theta \le 2\pi$

EX Evaluate the given double integral over the disk D given by $x^2+y^2 \leq 4$

$$\iint_D (x^2+y^2)\,dA$$



$x^2+y^2 \leq 4$

Polar coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} x^2+y^2=r^2$$

$x^2+y^2=r^2=4 \Rightarrow r=2$

$0 \leq r \leq 2$

$$\boxed{\begin{gathered} dA = r.dr\,d\theta \\ \text{OR} \\ dA = r\,d\theta\,dr \end{gathered}}$$

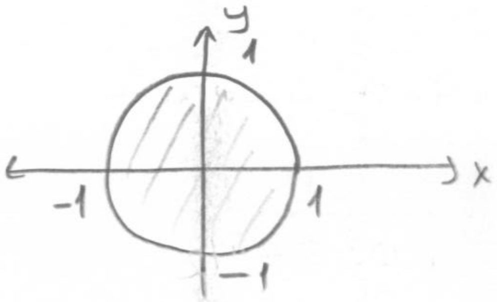
$$\iint_D (x^2+y^2)\,dA = \iint r^2.r.dr\,d\theta$$

$$\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 \,d\theta = 4\int_0^{2\pi} d\theta$$

$$= 4\theta\Big|_0^{2\pi} = 8\pi$$

Ex) Evaluate the given double integral over the disk D given by $x^2+y^2 \leq 1$

$$\iint_D \sqrt{x^2+y^2}\, dA$$



$x^2+y^2 \leq 1$

Polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow x^2+y^2 = r^2$

$r^2 = 1 = x^2+y^2 \Rightarrow r = 1$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

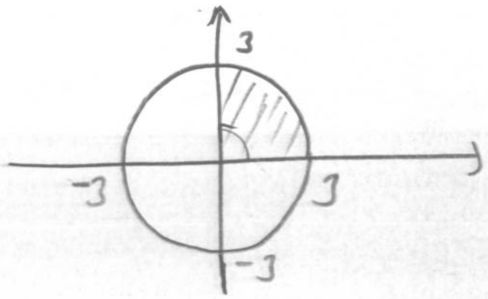
$\boxed{dA = r\,dr\,d\theta}$

$$\iint \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta = \frac{1}{3}\,\theta\Big|_0^{2\pi} = \frac{2\pi}{3}$$

(EX) Evaluate the given double integral

$\iint_{\theta} e^{x^2+y^2}\, dA$ over the quater-disk θ given by $x \geq 0, y \geq 0$ and $x^2+y^2 \leq 9$



$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\}$ the first quadrant

$x^2+y^2 \leq 9$, $0 \leq \theta \leq \pi/2$

Polar coordinates

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow x^2+y^2 = r^2$

$x^2+y^2 = r^2 = 9 \Rightarrow r = 3$

$0 \leq r \leq 3$

$\boxed{dA = r.\,dr.\,d\theta}$

$$\iint_{\theta} e^{x^2+y^2}\, dA = \int_0^{\pi/2}\int_0^3 e^{r^2}\, r\,dr\,d\theta \qquad r^2 = u \qquad r\,dr = du/2$$

when $r = 0$ $u = 0^2 = 0$

" $r = 3$ $u = 3^2 = 9$

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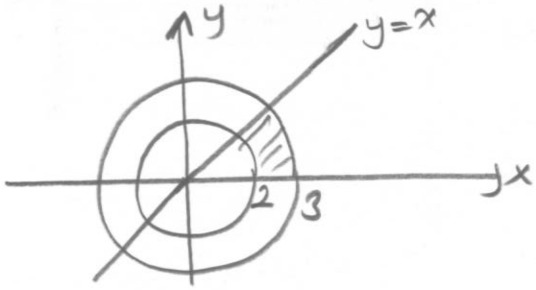
$$=\int_0^{\pi/2}\int_0^9 e^u \frac{1}{2}\,du\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2} e^u \Big|_0^9 d\theta=\frac{1}{2}\int_0^{\pi/2}(e^9-e^0)\,d\theta$$

$$=\frac{1}{4}(e^9-1)\pi \;//$$

EX Evaluate the double integral $\iint_R \frac{y^2}{x^2}\,dA$ where R is $4\le x^2+y^2\le 9$ in the first quadrant and below the line $y=x$



Polar coordinates

$$\left.\begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \end{aligned}\right\} \Rightarrow x^2+y^2=r^2$$

$$\left.\begin{aligned} x^2+y^2=r^2=4 &\Rightarrow r=2 \\ x^2+y^2=r^2=9 &\Rightarrow r=3 \end{aligned}\right\} 2\le r\le 3$$

$y=x \iff r\sin\theta=r\cos\theta$

$\tan\theta=1$

$\theta=\pi/4$

$0\le\theta\le\pi/4$

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$$\iint_R \frac{y^2}{x^2}\,dA = \int_0^{\pi/4}\int_2^3 \frac{(r\sin\theta)^2}{(r\cos\theta)^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_2^3 r\cdot \tan^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \left.\frac{r^2}{2}\right|_2^3 d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \left(\frac{3^2}{2} - \frac{2^2}{2}\right) d\theta$$

$$= \frac{5}{2}\int_0^{\pi/4} \tan^2\theta\, d\theta$$

$$= \frac{5}{2}\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$$

$$= \frac{5}{2}\left[\tan\theta - \theta\right]\Big|_0^{\pi/4}$$

= TRIPLE INTEGRAL =

* For a bounded function $f(x,y,z)$ defined on a rectangular box B the triple integral of over B is given by

$$\iiint_B f(x,y,z)\,dV \quad \text{or} \quad \iiint_B f(x,y,z)\,dx\,dy\,dz$$

$\Rightarrow$ If B is given by $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$, $c_1 \leq z \leq c_2$

then

$$\int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$$

$\Rightarrow$ If $f(x,y,z) = 1$ on the domain D then the triple integral gives the volume of D

$$\text{Volume of } D = \iiint_D dV$$

$dV = dx\,dy\,dz$ or

$= dx\,dz\,dy$ "

$= dy\,dx\,dz$ "

$= dy\,dz\,dx$ "

$= dz\,dx\,dy$ "

$= dz\,dy\,dx$ "

Ex Evaluate the triple integral $\iiint\limits_R (1+2x-3y)\,dV$ over the box $-1 \le x \le 1$, $-2 \le y \le 2$, $-3 \le z \le 3$

$$\iiint\limits_R (1+2x-3y)\,dV$$

$$= \int_{-1}^{1}\int_{-2}^{2}\underbrace{\int_{-3}^{3} (1+2x-3y)\,dz}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-2}^{2} (1+2x-3y)z\Big|_{-3}^{3}\,dy\,dx$$

$$= \int_{-1}^{1}\underbrace{\int_{-2}^{2} 6(1+2x-3y)\,dy}\,dx$$

$$= \int_{-1}^{1} 6\left(y^2+2xy-\frac{3y^2}{2}\right)\Big|_{-2}^{2}\,dx$$

$$= \int_{-1}^{1} 6\left[(2-(-2))+2x(2-(-2))-\frac{3}{2}(2^2-(-2)^2)\right]dx$$

$$= \int_{-1}^{1} 6(4+8x)\,dx = 6\left(4x+\frac{8x^2}{2}\right)\Big|_{-1}^{1}$$

$$= 6\cdot\left[4(1-(-1))+4(1^2-(-1)^2\right] = 48$$

EX Evaluate triple integral $\iiint_B xyz\,dV$ where B is the Box given

$0 \le x \le 1,\ -2 \le y \le 0,\ 1 \le z \le 4$

$$\iiint_B xyz\,dV =$$

$$= \int_0^1 \int_{-2}^0 \underbrace{\int_1^4 xyz\,dz}\,dy\,dx$$

$$= \int_0^1 \int_{-2}^0 xy\frac{z^2}{2}\Big|_1^4\,dy\,dx$$

$$= \int_0^1 \int_{-2}^0 \frac{1}{2}xy\,(4^2-(-1)^2)\,dy\,dx$$

$$= \int_0^1 \underbrace{\int_{-2}^0 \frac{15}{2}\,xy\,dy}\,dx = \int_0^1 \frac{15}{2}x\frac{y^2}{2}\Big|_{-2}^0\,dx$$

$$= \int_0^1 \frac{15}{4}x\,[0^2-(-2)^2]\,dx$$

$$= \int_0^1 -15x\,dx = -15\frac{x^2}{2}\Big|_0^1 = -\frac{15}{2}$$

EX (2011, final) Evaluate the triple integral

$$\iiint_R xy^2 e^{-xyz}\, dV$$ over the cube $0 \leq x, y, z \leq 1$

——o——

$$\iint \int_0^1 xy^2 e^{-xyz}\, dz\, dA$$

dydx or dxdy

$$= \iint \frac{xy^2 e^{-xyz}}{-xy} \Big|_0^1 dA$$

$$= \iint -y\,(e^{-xy1} - e^{-xy0})\, dA$$

$$= \iint -y\,(e^{-xy} - 1)\, dA$$

$$= \iint (y - y\, e^{-xy})\, dA = \int_0^1 \int_0^1 (y - y\, e^{-xy})\, dx\, dy$$

$$= \int_0^1 \left[yx - y \cdot \frac{e^{-xy}}{-y} \right] \Big|_0^1 dy$$

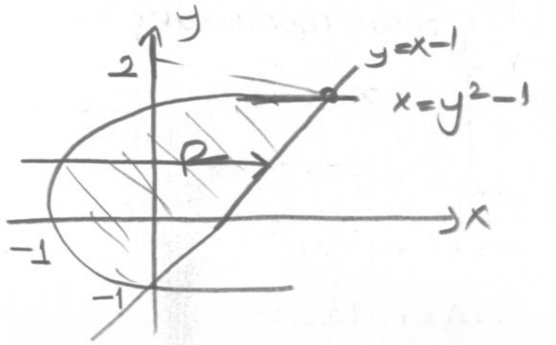
$$= \int_0^1 \left[y + e^{-y} - 1 \right] dy = \frac{y^2}{2} - e^{-y} - y \Big|_0^1 =$$

$$= \frac{1}{2} - (e^{-1} - e^{0}) - 1$$

$$= \frac{1}{2} - \frac{1}{e}$$

Ex) Evaluate the double integral $\iint_R (x+xy)\,dA$ where R is bounded by the curves $x=y^2-1$ and $y=x-1$

—— o ——



$$\left.\begin{array}{l} x=y^2-1 \\ x=y+1 \end{array}\right\} \Rightarrow y^2-1=y+1$$

$$y^2-y-2=0$$

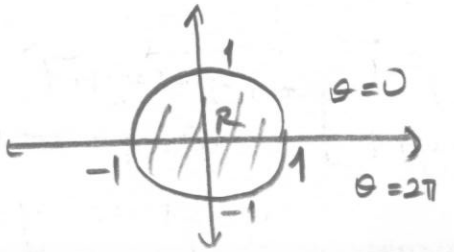
$$y=-1 \text{ or } y=2$$

$$\iint_R (x+xy)\,dA = \int_{-1}^{2}\underbrace{\int_{y^2-1}^{y+1} (x+xy)\,dx}\,dy$$

$$=\int_{-1}^{1} \left(\frac{x^2}{2}+\frac{x^2}{2}y\right)\Bigg|_{y^2-1}^{y+1} dy$$

$$=\int_{-1}^{1} \left[\frac{1}{2}(y+1)^2-(y^2-1)^2\right]+\frac{1}{2^y}\left[(y+1)^2-(y-1)^2\right]dy$$

Ex) Evaluate the double integral $\iint_R e^{-x^2-y^2}\,dA$ where R is the disk $x^2+y^2 \le 1$



disk $\rightarrow$ $\begin{cases} \text{polar coordinates} \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow x^2+y^2=r^2$

$dA = r.dr.d\theta$

$$\iint_R e^{-[x^2+y^2]}\,dA = \int_0^{2\pi}\int_0^1 e^{-r^2}\,r\,dr\,d\theta$$

$-r^2 = u \qquad r=0 \rightarrow u=0^2=0$

$r\,dr = -\frac{du}{2} \qquad r=1 \rightarrow u=-1^2=-1$

$$= \int_0^{2\pi}\int_0^{-1} e^u \cdot -\frac{du}{2}\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}e^u \Big|_{-1}^{0} d\theta = \int_0^{2\pi} -\frac{1}{2}(e^0-e^{-1})\,d\theta$$

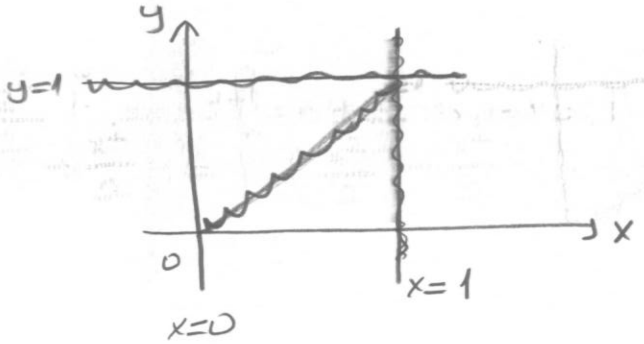
$$= \frac{1}{2}\left(1-\frac{1}{e}\right)\int_0^{2\pi} d\theta = \pi\left(1-\frac{1}{e}\right)$$

EX Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3}\,dy\,dx$$

$$R: \begin{cases} 0 \le x \le 1 \\ \sqrt{x} \le y \le 1 \end{cases}$$

$$\int_0^1 \int_0^{y^2} \frac{3}{4+y^3}\,dx\,dy$$

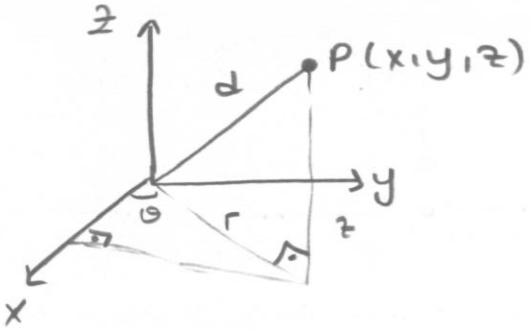


$$= \int_0^1 \frac{3}{4+y^3}\, x \Big|_0^{y^2} dy = \int_0^1 \frac{\overbrace{3y^2\,dy}^{du}}{\underbrace{4+y^3}_{u}}$$

$$= \ln|4+y^3| \Big|_0^1$$

$$= \ln 5 - \ln 4 = \ln \frac{5}{4}$$

Cylindirical Coordinates



The cylindirical coordinates of a point

cartesion coordinates $P=(x,y,z)$

cylindrical " $P=[r,\theta,z]$

** $\left[\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{array}\right.$ ** $[dV=r.dr.d\theta dz]$

(EX) Evaluote

$$\iiint_R (x^2+y^2)\,dV$$ over the first octant

region bounded the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and the planes $z=0, z=1, x=0$ and $x=y$

$$\begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{bmatrix} \Rightarrow x^2+y^2=r^2$$

$$r^2 = 1 \longrightarrow r = 1$$

$$r^2 = 4 \rightarrow r = 2$$

$$1 \leq r \leq 2$$

$$\left.\begin{array}{l} z = z = 0 \\ z = z = 1 \end{array}\right\} \quad 0 \leq z \leq 1$$

$$\left.\begin{array}{l} x = 0 \\ x = r\cos\theta \end{array}\right\} \Rightarrow \begin{array}{l} r\cos\theta = 0 \\ r \neq 0 \quad \theta = \pi/2 \end{array}$$

$$x = y \Rightarrow \cancel{r}\cos\theta = \cancel{r}\sin\theta$$

$$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1 \quad , \theta = \pi/4$$

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

$$\iiint_R (x^2+y^2)\,dV = \int_0^1 \int_{\pi/4}^{\pi/2} \int_1^2 r^2\, r\,dr\,d\theta\,dz = \int_0^1 \int_{\pi/4}^{\pi/2} \frac{r^4}{4}\Big|_1^2 \,d\theta\,dz$$

$$= \int_0^1 \int_{\pi/4}^{\pi/2} \frac{1}{4}(2^4-1^4)\,d\theta\,dz = \frac{15}{4}\int_0^1 \int_{\pi/4}^{\pi/2} d\theta\,dz = \frac{15}{4}\int_0^1 \theta \Big|_{\pi/4}^{\pi/2} dz$$

$$= \frac{15}{4}\int_0^1 [\pi/2 - \pi/4]\,dz = \frac{15}{4}\cdot\frac{\pi}{4}\int_0^1 dz = \frac{15\pi}{16}$$

EX Use the lagrange multiplier method to find greatest and least distance from the point $(2,1,-2)$ to sphere with equation $x^2+y^2+z^2=1$

Let (x,y,z) be the point on the given curve. Then the distance from $(2,1,-2)$ to (x,y,z)

$$d=\sqrt{(x-2)^2+(y-1)^2+(z+2)^2}$$

$$d^2=(x-2)^2+(y-1)^2+(z+2)^2$$
$$=f(x,y,z)$$

$$x^2+y^2+z^2=1 \Rightarrow x^2+y^2+z^2-1=0=g(x,y,z)$$

$$L(x,y,z,\lambda)=f(x,y,z)+\lambda(x^2+y^2+z^2-1)$$
$$=(x-2)^2+(y-1)^2+(z+2)^2+\lambda(x^2+y^2+z^2-1)$$

For CP:

$$\frac{\partial L}{\partial x}=2(x-2)+\lambda 2x=0$$

$$\frac{\partial L}{\partial y}=2(y-1)+\lambda 2y=0$$

$$\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1=0$$

$x^2+y^2+z^2-1=0 \Rightarrow$

$$\left[\frac{2}{1+\lambda}\right]^2+\left[\frac{1}{1+\lambda}\right]^2+\left[\frac{-2}{1+\lambda}\right]^2=1$$

$$=\left(\frac{1}{1+\lambda}\right)^2[4+1+4]=1$$

$$\frac{1}{(1+\lambda)^2}=\frac{1}{9} \qquad (1+\lambda)^2=9$$

$1+\lambda=3 \Rightarrow \lambda=2$

or

$1+\lambda=-3$

$\lambda=-4$

Let $\lambda=2$. Then

$$x=\frac{2}{1+\lambda}=\frac{2}{1+(+2)}=\frac{2}{3}, \quad y=\frac{1}{1+\lambda}=\frac{1}{1+(+2)}=\frac{1}{3}$$

$$z=\frac{-2}{1+\lambda}=\frac{-2}{1+(+2)}=\frac{-2}{3} \qquad \left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$$

Let $\lambda=-4$, Then we get $\left(\frac{-2}{3},\frac{-1}{3},\frac{2}{3}\right)$

For $\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right) \Rightarrow d= - - -$

For $\left(\frac{-2}{3},\frac{-1}{3},\frac{2}{3}\right) \Rightarrow d= - - -$

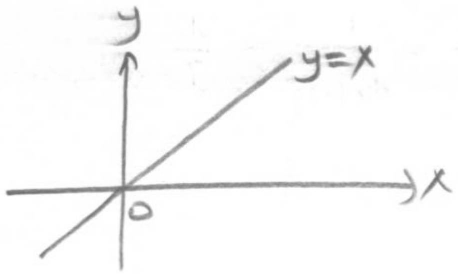
EX Given the function

$$f(x,y)=\begin{cases} \dfrac{2xy^2}{x^2+y^4} & \text{if} \quad (x,y)\neq(0,0) \\ 0 & \text{if} \quad (x,y)=(0,0) \end{cases}$$

Is f continuous at $(0,0)$?

$f(x,y)$ is continuous at $(a,b) \iff \lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$

$$\lim_{(x,y)\to(a,b)} f(x,y)=\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$$



$(x,y)\to(0,0)$ along the line $y=0$

$$f(x,y)=\frac{2xy^2}{x^2+y^4} \Rightarrow f(x,0)=\frac{2.x.0^2}{x^2+0^4}=\frac{0}{x}\to 0 \quad \text{as } x\to 0$$

$(x,y)\to(0,0)$ along the line $x=y$

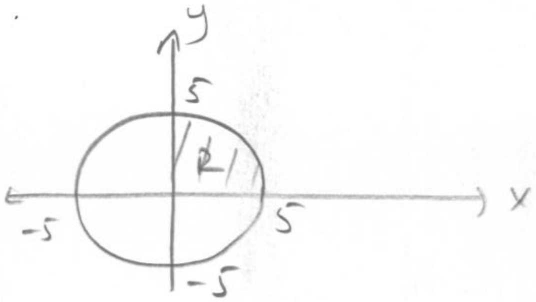
$$f(y,y)=\frac{2y.y^2}{y^2+y^4}=\frac{2y}{1+y^2}\to 0 \quad \text{as } y\to 0$$

$$f(y^2,y)=\frac{2y^2y^2}{(y^2)^2+y^4}=\frac{2y^4}{2y^4}=1 \quad \text{as} \quad y\to 0$$

$0\neq 1 \Rightarrow$ the limit does not exist so $f(x,y)$ is discontinuous at $(0,0)$

EX) Evaluate the given double integral over quarter-disk R given by $x \geq 0, y \geq 0$ and $x^2+y^2 \leq 25$

$$\iint_R e^{2x^2+2y^2}\, dA$$



Polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2=r^2 \end{cases}$

$x^2+y^2 = r^2 = 25 \rightarrow r=5$

$0 \leq r \leq 5$

$0 \leq \theta \leq \pi/2$

$\boxed{dA = r\,dr\,d\theta}$

$$\iint_R e^{2(x^2+y^2)}\, dA = \int_0^{\pi/2}\underbrace{\int_0^5 e^{2r^2}\, r\,dr}_{}\,d\theta$$

$2r^2 = u \rightarrow 4r\,dr = du$

when $r=0$, $u=0$

" $r=5$, $u=2.5^2=50$

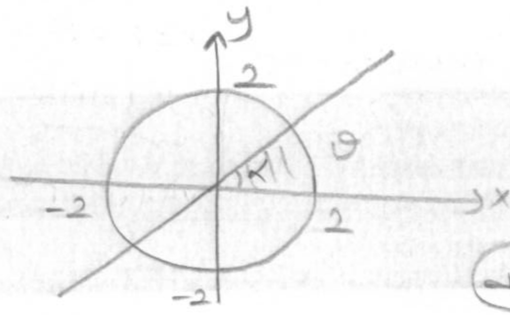
$$= \int_0^{\pi/2} \frac{1}{4} e^u \Big|_0^{50} d\theta = \int_0^{\pi/2} \frac{1}{4}\left[e^{50}-e^0\right] d\theta$$

$$= \frac{1}{4}(e^{50}-1)\,\theta \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}\cdot(e^{50}-1) \quad //$$

EX Evaluate

$\iint_R (x+y)dA$ where R is the region in the first quadrant lying inside the disk $x^2+y^2 \leq 4$ and under the line $y=\sqrt{3}x$



disk polar coord

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}} \Rightarrow x^2+y^2=r^2$$

$x^2+y^2=r^2=4 \rightarrow r=2 \quad , 0 \leq r \leq 2$

$y=\sqrt{3}x \Rightarrow r\cos\theta = \sqrt{3}.r.\sin\theta$

$\tan^{-1}\frac{1}{\sqrt{3}} = \theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \pi/3$

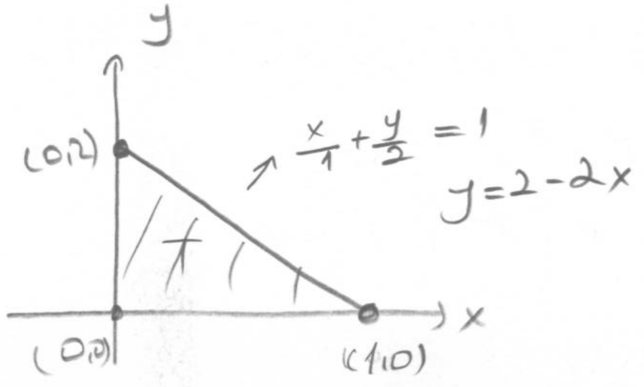
$$\iint_R (x+y)\,dA = \int_0^{\pi/3}\int_0^2 [r\cos\theta + r\sin\theta]\,r\,dr\,d\theta$$

$$= \int_0^{\pi/3}\int_0^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/3} \left(\frac{r^3}{3}\Big|_0^2\right)(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{8}{3}(\sin\theta - \cos\theta)\Big|_0^{\pi/3}$$

EX Evaluate $\iint_T (x-3y)\,dA$, where T is the triangle with vertices (0,0), (1,0) and (0,2)



$dA = dx\,dy$ or $dA = dy\,dx$

$$\iint_T (x-3y)\,dA = \int_0^1\int_0^{2-2x} (x-3y)\,dy\,dx$$

$$= \int_0^1 \left[xy - 3\frac{y^2}{2}\right]\Big|_0^{2-2x} dx$$

$$= \int_0^1 \left[x(2-2x) - \frac{3}{2}(2-2x)^2\right] dx$$

$$= \int_0^1 \left[2x - x^2 - \frac{3}{2}(4-8x+4x^2)\right] dx$$

$$= \int_0^1 \left[-7x^2 + 14x - 6\right] dx = -\frac{7}{3}x^3 + \frac{14x^2}{2} - 6x\Big|_0^1$$

$$= -\frac{7}{3} + 7 - 6 = -\frac{4}{3}$$