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IZMIR UNIVERSITY OF ECONOMICS

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Faculty of Arts and Sciences

Term	: 23-24 Fall
Course ID	: MATH 250
Exam	: Midterm Exam
Date	: 22.11.2023
Duration	: 75 minutes
Instructor	:

Full Name : ....

Student ID : ....

Classroom : .... **Section** : ....

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Declaration

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Signature of Student:

Question	1	2	3	4
Score	/	/	/	/
Total	/100			

1. (a) Describe all solutions of $Ax = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 3 & -1 & -4 \\ 2 & 6 & 4 & -8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -1 & -4 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

pivot

pivot

x_2 and x_4 are free variables

$6x_3 = 0 \Rightarrow x_3 = 0$

$x_1 + 3x_2 - x_3 - 4x_4 = 0 \Rightarrow x_1 = -3x_2 + 4x_4$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + 4x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = \underset{a}{x_2} \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + \underset{b}{x_4} \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \Rightarrow x = av_1 + bv_2$$

(b) Find conditions on a, b and c such that the system of linear equations

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$$\begin{aligned} 2x - y + z &= a \\ x + y + 2z &= b \\ 3y + 3z &= c \end{aligned}$$

has

(i) no solution,

(ii) exactly one solution and

(iii) an infinite number of solutions.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & 3 & 3 & c \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 2 & -1 & 1 & a \\ 0 & 3 & 3 & c \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & -3 & -3 & a-2b \\ 0 & 3 & 3 & c \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & b \\ 0 & -3 & -3 & a-2b \\ 0 & 0 & 0 & c+a-2b \end{array}\right]$$

If $a - 2b + c \neq 0$, then no sol!

If $a - 2b + c = 0$, then inf many sol.

It is impossible to get unique sol, since there are only two pivots.

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2. (a) Are the vectors

$$S = \left\{ \begin{bmatrix} 7 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 15 \\ 3 \\ -12 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix} \right\}$$

linearly independent? Do they span $\mathbb{R}^3$? Explain your reason.

$$\begin{bmatrix} 7 & 15 & 6 \\ 1 & 3 & -2 \\ -5 & -12 & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 3 & -2 \\ 7 & 15 & 6 \\ -5 & -12 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 5R_1]{R_2 \to R_2 - 7R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 20 \\ 0 & 3 & -4 \end{bmatrix}$$

pivot, pivot

$$\downarrow R_3 \to R_3 + \tfrac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 20 \\ 0 & 0 & 6 \end{bmatrix}$$

pivot

the given vectors are linearly independent since each column has a pivot position and hence they span $\mathbb{R}^3$.

(b) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with the rule

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ 2y + 3z \\ x - y + z \end{pmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i. Write the matrix of the transformation.

ii. Is T 1-1? Show your work.

iii. Is T onto? Show your work.

i) $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

ii) T is one-to-one if $Ax = 0$ has only the trivial solution.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\downarrow R_3 \to R_3 - \tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -\frac{5}{2} & 0 \end{array}\right]$$

$-\frac{5}{2}z = 0 \Rightarrow z = 0$

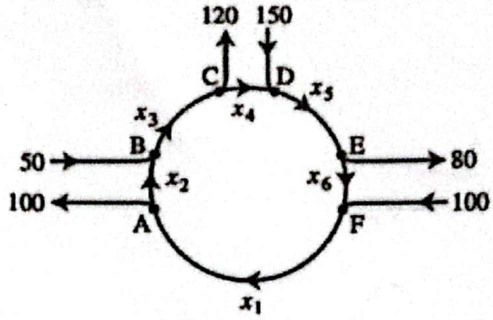
$2y + 3z = 0 \Rightarrow y = 0$

$x - y + z = 0 \Rightarrow x = 0$

So the transformation is one-to-one

iii) when we solve the system $Ax = b$, each b has at least one solution x, T is onto.

3. (a) Write the corresponding linear system to the traffic network. Do not solve!



A: $x_1 = 100 + x_2$

B: $x_2 + 50 = x_3$

C: $x_3 = 120 + x_4$

D: $x_4 + 150 = x_5$

E: $x_5 = 80 + x_6$

F: $x_6 + 100 = x_1$

$$\left.\begin{aligned} x_1 - x_2 &= 100 \\ -x_2 + x_3 &= 50 \\ x_3 - x_4 &= 120 \\ -x_4 + x_5 &= 150 \\ x_5 - x_6 &= 80 \\ x_1 - x_6 &= 100 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \\ 120 \\ 150 \\ 80 \\ 100 \end{bmatrix}$$

(b) **(i)** If a matrix A is 5×3 and the product matrix AB is 5×7, what is the size of the matrix B?

(ii) If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $Dx = b$? Explain your answer.

(iii) How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

(iv) Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

i) $A_{5\times3} \cdot B = AB_{5\times7}$ then $B: 3\times7$

ii) If all columns of D are linearly independent then each column has a pivot posrtion and hence $Dx = b$ has a unique solution

iii) since there are 5 columns, all of them should have pivot posrtion in order to be linearly independent

iv) If columns do not span $\mathbb{R}^5$, then there is at least one linearly dependent column vector. So, A can not be invertible

4. (a) Find the inverse of

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 3 & -1 & 4 \\ 6 & -3 & 11 \end{bmatrix}$$

if it exists. State whether the column vectors of the given matrix linearly independent or not.

pivot

$$\begin{bmatrix} 3 & -2 & 7 \\ 3 & -1 & 4 \\ 6 & -3 & 11 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 3 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot

non pivot

Since third row does not have the pivot position A is not invertible. The column vectors are linearly dependent.

(b) Find an LU factorization of the given matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + R_3} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{2}{3} R_2}$$

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}$$