25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 250 Linear Algebra for Engineers

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

MIDTERM EXAM 23.11.2022

Name: ....

Student No: ....

KEY

CamScanner ile tarandı

1. (a) Let $u = \begin{bmatrix}4\\-7\\-4\end{bmatrix}$, $A = \begin{bmatrix}1 & -6 & 4\\0 & 6 & 2\\-2 & 12 & -8\end{bmatrix}$. Is u in the plane spanned by the columns of A. Does the columns of A span R^3? Why or why not?

$$c_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+c_2\begin{bmatrix}-6\\6\\12\end{bmatrix}+c_3\begin{bmatrix}4\\2\\-8\end{bmatrix}=\begin{bmatrix}4\\-7\\-4\end{bmatrix} \rightarrow \begin{bmatrix}1&-6&4\\0&6&2\\-2&12&-8\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}4\\-7\\-4\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&-6&4&4\\0&6&2&-7\\-2&12&-8&-4\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&-6&4&4\\0&6&2&-7\\0&0&0&4\end{array}\right] \rightarrow \text{no sol.} \quad u \notin \text{span}(\text{Col}1, \text{Col}2, \text{Col}3).$$

Since there is one vector $u \in \mathbb{R}^3$ which can't be spaned by colums of A, $\text{span}\{\text{col}1, \text{col}2, \text{col}3\} \neq \mathbb{R}^3$.

(b) Express the solution space of the homogeneous linear system

$$\begin{aligned} x - 2y + z - w &= 0\\ -x + y + 2w &= 0\\ 2x - 3y + z - 3w &= 0 \end{aligned}$$

by using suitable number of free parameters. Also, represent the solution space by spanning set of vectors.

$$\begin{bmatrix}1&-2&1&-1\\-1&1&0&2\\2&-3&1&-3\end{bmatrix} \longrightarrow \begin{bmatrix}1&-2&1&-1\\0&-1&1&+1\\0&1&-1&-1\end{bmatrix} \rightarrow \begin{bmatrix}1&-2&1&-1\\0&-1&1&1\\0&0&0&0\end{bmatrix}$$

We need 2-free parameters $z = k_1$ / $w = k_2$ where $k_1, k_2 \in \mathbb{R}$.

then $-y + k_1 + k_2 = 0 \rightarrow \boxed{y = k_1 + k_2}$

$x - 2(k_1 + k_2) + k_1 - k_2 = 0 \rightarrow x - k_1 - 3k_2 = 0$

$\boxed{x = k_1 + 3k_2}$

$$\text{sol space} = \left\{ \begin{bmatrix}k_1+3k_2\\k_1+k_2\\k_1\\k_2\end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix} \right\}$$

CamScanner ile tarandı

2. (a) Set up a system of linear equations to represent the network in the following figure. Express the system in matrix-vector form. Do NOT solve the system.

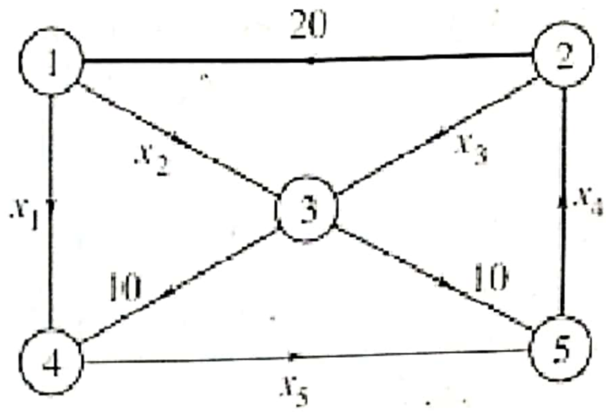
$x_1 + x_2 = 20$

$x_3 - x_4 = -20$

$x_2 + x_3 = 20$

$x_1 - x_5 = -10$

$-x_4 + x_5 = -10$



The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right] \xrightarrow{-R_1+R_4 \to R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & -1 & 0 & 0 & -1 & -30 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right] \xrightarrow{R_2 \leftrightarrow -R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & +1 & 30 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & +1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right] \xrightarrow{-R_3+R_4 \to R_4} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & +1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & +1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_5 = t$, $x_4 = t+10$, $x_3 = t-10$, $x_2 = -t+30$, $x_1 = t-10$

(b) The population of a city in 2000 was 800,000 while the population of the suburbs of that city in 2000 was 900,000. Suppose that demographic studies show that each year about %7 of the city's population moves to the suburbs (and %93 stays in the city), while %2 of the suburban population moves to the city (and %98 stays in the suburbs). Compute the population of the city and suburbs in 2002. For simplicity, ignore other influences on the population such as births, deaths, and migration into and out of the city/suburban region.

Migration matrix $M = \begin{array}{cc} C & S \end{array}$ $\begin{array}{c} \\ \end{array}\left[\begin{array}{cc} 0{,}93 & 0{,}02 \\ 0{,}07 & 0{,}98 \end{array}\right]\begin{array}{c} C \\ S \end{array}$, $2000 \to \begin{bmatrix} C \\ S \end{bmatrix} = \begin{bmatrix} 800k \\ 900k \end{bmatrix}$

$2001 \to \begin{bmatrix} 0{,}93 & 0{,}02 \\ 0{,}07 & 0{,}98 \end{bmatrix}\begin{bmatrix} 800k \\ 900k \end{bmatrix} = \begin{bmatrix} 762k \\ 938k \end{bmatrix}$

$2002 \to \begin{bmatrix} 0{,}93 & 0{,}02 \\ 0{,}07 & 0{,}98 \end{bmatrix}\begin{bmatrix} 762.000 \\ 938.000 \end{bmatrix} = \begin{bmatrix} 727{,}420 \\ 972{,}580 \end{bmatrix}$

CamScanner ile tarandı

3. (a) Determine if the specified linear transformation is one-to-one and onto. Justify each answer.

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 2x_3, x_2 - 9x_3)$$

$\ker T = \{\vec{0}\} \Leftrightarrow T$ is 1-1.

$\left.\begin{array}{l} x_1 - 5x_2 + 2x_3 = 0 \\ x_2 - 9x_3 = 0 \end{array}\right\}$ Cant have trivial sol!

not 1-1.

$\text{Range}(T) = x_1(1,0) + x_2(-5,1) + x_3(2,-9)$

$= \text{Span}\{(1,0), (-5,1), (2,-9)\} = \mathbb{R}^2$ ✓

$\therefore$ T is onto.

(b) Consider the linear transformations $L_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $L_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $L_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as

$$L_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ y \\ -x + y \end{pmatrix}, L_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y + z \\ -x \end{pmatrix}, L_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ -y + z \\ 2z \end{pmatrix}$$

with matrices A_1, A_2 and A_3, respectively. Write A_1, A_2 and A_3. Then, evaluate $2A_1A_2 - A_3$.

$$A_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}_{2\times 3} \quad A_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 . A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

$$2A_1 A_2 = \begin{bmatrix} 4 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix}, \quad -A_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$2A_1 A_2 - A_3 = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 1 & -1 \\ -2 & -2 & -4 \end{bmatrix}$$

CamScanner ile tarandı

4. (a) Find the inverse of the coefficient matrix and solve the system by using the inverse matrix.

$$\begin{aligned} 4x + 6y &= 16 \\ 2x + 7y &= 24 \end{aligned}$$

$A = \begin{bmatrix} 4 & 6 \\ 2 & 7 \end{bmatrix} \rightarrow \det A = 28 - 12 = 16$

$A^{-1} = \frac{1}{16} \begin{bmatrix} 7 & -6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 7/16 & -3/8 \\ -1/8 & 1/4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7/16 & -3/8 \\ -1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 16 \\ 24 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$

(b) Find the inverse of

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 3 & -1 & 4 \\ 6 & -3 & 11 \end{bmatrix}$$

if it exists. State whether the column vectors of the given matrix linearly independent or not.

i) The inverse does not exist.

ii) Columns are not linearly independent, that is

Explanation (i): $\begin{bmatrix} 3 & -2 & 7 \\ 3 & -1 & 4 \\ 6 & -3 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

2-pivots!

ii) $c_1 \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 & -2 & 7 \\ 3 & -1 & 4 \\ 6 & -3 & 11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The matrix has 2-pivot columns. The system can't have trivial sol.

CamScanner ile tarandı