25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 250 Linear Algebra for Engineers

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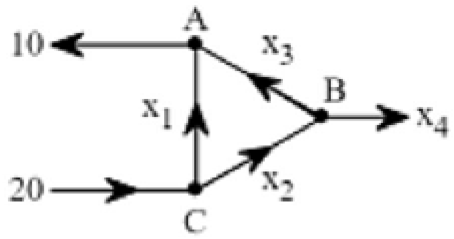
KEY FOR MIDTERM EXAM 12.05.2022

Name: ....

Student No: ....

1. a. (15pt)Write the linear system corresponding to the network shown in the figure, and then solve it.



$$
\begin{aligned}
10 + x_4 &= 20 \\
x_1 + x_2 &= 20 \\
x_1 + x_3 &= 10 \\
-x_2 + x_3 + x_4 &= 0
\end{aligned}
$$

This implies $x_4 = 10$ and

$$
\begin{aligned}
x_1 + x_2 &= 20 \\
x_1 + x_3 &= 10 \\
-x_2 + x_3 &= -10
\end{aligned}
$$

Then $\begin{bmatrix} 1 & 1 & 0 & 20 \\ 1 & 0 & 1 & 10 \\ 0 & -1 & 1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & -1 & 1 & -10 \\ 0 & -1 & 1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & -1 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We get $x_2 = x_3 + 10, x_1 = 10 - x_3,\ x_4 = 10.$

b.(10pt) Assuming that the flows are all nonnegative, what is the largest possible value for x_3?

Since $x_i \geq 0$ for $i = 1, 2, 3, 4$ the maximum value for x_3 is 10.

2. a. (12pt)Let $v_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$ be vectors in $\mathbb{R}^3$. Find the value(s) of h for which the vectors are linearly dependent By using the given vectors

we write the linear combination

$$c_1 \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and write the matrix

$\begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 4 & 7 & h \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 \\ 0 & -2 & 0 \\ 0 & -1 & h+4 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & h+4 \end{bmatrix}$. For linear independence of the vectors, we must have $h \neq -4$.

b. (13pt) If T is defined by T(x)=Ax, find a vector x whose image under T is **b,** and determine whether **x** is unique. Let $A = \begin{bmatrix} 1 & -4 & -7 \\ -2 & 1 & -7 \end{bmatrix}$ and **b**=$\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

We consider $T(x) = Ax$ and $Ax = b$, that is

$$\begin{bmatrix} 1 & -4 & -7 \\ -2 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

If we solve the system

$$\begin{bmatrix} 1 & -4 & -7 & -2 \\ -2 & 1 & -7 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & -7 & -2 \\ 0 & -7 & -21 & -7 \end{bmatrix},$$

then by 2nd row $-7x_2 = -7 + 21x_3$. Thus $x_2 = 1 - 3x_3$. By the first row

$$x_1 - 4x_2 - 7x_3 = -2$$

and

$$x_1 - 4(1 - 3x_3) - 7x_3 = x_1 + 5x_3 - 4 = -2.$$

We get

$$x_1 = 2 - 5x_3.$$

By setting $x_3 = k$, we establish the solution space as

$$S = \left\{ \begin{bmatrix} 2-5k \\ 1-3k \\ k \end{bmatrix}, k \in \mathbb{R} \right\}.$$

Therefore, x is not uniquely determined.

3. Consider the matrices given below:

$$A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ -3 & 1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 5 & -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & -3 \\ 2 & 7 \end{bmatrix}.$$

(a) (12 pt)Find $-2A + CB$.

$$-2A = \begin{bmatrix} -4 & 2 & 0 & -8 \\ 6 & -2 & -4 & 0 \end{bmatrix}, CB = \begin{bmatrix} 1 & -3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 5 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -12 & 1 & -5 \\ 2 & 41 & -11 & 16 \end{bmatrix}$$

$$-2A + CB = \begin{bmatrix} -3 & -10 & 1 & -13 \\ 8 & 39 & -15 & 16 \end{bmatrix}.$$

(b) (13 pt) Find $AB^T - 3C$.

$$B^T = \begin{bmatrix} 1 & 0 \\ 3 & 5 \\ -2 & -1 \\ 1 & 2 \end{bmatrix}, AB^T = \begin{bmatrix} 2 & -1 & 0 & 4 \\ -3 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 5 \\ -2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -4 & 3 \end{bmatrix},$$

$$-3C = \begin{bmatrix} -3 & 9 \\ -6 & -21 \end{bmatrix}$$

$$AB^T - 3C = \begin{bmatrix} 3 & 3 \\ -4 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 9 \\ -6 & -21 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -10 & -18 \end{bmatrix}.$$

4. (a) (13 pt) Write the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 3 & -1 \\ -1 & 3 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 2 & -1 \\ 4 & 3 & -1 \\ -1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 4 & 3/2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 11/2 \end{bmatrix} = U, \text{ also } \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \div$$

$$2 = \begin{bmatrix} 1 \\ 2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \div (-1) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}. \text{ Then } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix} \text{ and}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 11/2 \end{bmatrix}.$$

(b) (12 pt) Calculate determinant of the matrix A by using cofactor expansion.

$$\to \begin{bmatrix} 2 & 2 & -1 \\ 4 & 3 & -1 \\ -1 & 3 & 2 \end{bmatrix} \det A = 2\,(-1)^2 \begin{vmatrix} 3 & -1 \\ 3 & 2 \end{vmatrix} + 2\,(-1)^3 \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} - 1\,(-1)^4 \begin{vmatrix} 4 & 3 \\ -1 & 3 \end{vmatrix} =$$

$18 - 14 - 15 = -11.$