30 points	15 points	30 points	25 points	100 points
1	2	3	4	**Total**

MATH 250 Linear Algebra for Engineers

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MIDTERM EXAM 12.12.2021

Name: ....#KEY#..............................

Student No: ....

1. For a given system of linear equations

$$\begin{aligned} 2x + y - z &= 1, \\ -x - 2y + 3z &= 0, \\ 5x + 4y + \alpha z &= -2. \end{aligned}$$

(a) Write the augmented matrix of the system above. Use row operations to reduce this augmented matrix to its echelon form.

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ -1 & -2 & 3 & 0 \\ 5 & 4 & \alpha & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 1 & -1 & 1 \\ 5 & 4 & \alpha & -2 \end{bmatrix} \xrightarrow[5R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2}$$

$$\begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & -6 & \alpha+15 & -2 \end{bmatrix} \xrightarrow{-2R_2 + R_3 \to R_3} \begin{bmatrix} -1 & -2 & 3 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & \alpha+5 & -4 \end{bmatrix}$$

Echelon form of the Augmented matrix.

(b) Determine for which values of α the system above has a unique solution.

If $\alpha + 5 = 0$, that is if $\alpha = -5$, then the system has no solution.

If $\alpha + 5 \neq 0$, that is if $\alpha \neq -5$, then the system has unique solution.

2. Determine if the vectors

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$$

are linearly independent. Explain why.

Let $A = [a_1 \ a_2 \ a_3] = \begin{bmatrix} 1 & -2 & 9 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

then the homogeneous equation $Ax = 0$ has augmented matrix

$$\begin{bmatrix} 1 & -2 & 9 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad \left.\begin{matrix} x_3 = 0 \\ x_2 = 0 \\ x_1 = 0 \end{matrix}\right\}$$

Only trivial solution exists. So a_1, a_2, a_3 are linearly independent

OR: Let $A = [a_1 \ a_2 \ a_3] = \begin{bmatrix} 1 & -2 & 9 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ is row equivalent to the identity matrix I_3

then A is invertible and by Invertible Matrix Theorem, the Columns of A are linearly independent.

3. Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 8 \\ 2 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -16 & 1 & 8 \\ -11 & 1 & 5 \\ 2 & 0 & -1 \end{bmatrix}$.

(a) Compute $2A^\top - B$, where $^\top$ denotes the transpose of the matrix.

(b) Show that $B = A^{-1}$.

(c) Determine if the columns of A span R^3

(a) $A^T = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -2 \\ 3 & 8 & 5 \end{bmatrix}$, $2A^T - B = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 0 & -4 \\ 6 & 16 & 10 \end{bmatrix} - \begin{bmatrix} -16 & 1 & 8 \\ -11 & 1 & 5 \\ 2 & 0 & -1 \end{bmatrix}$

$\Rightarrow$ $2A^T - B = \begin{bmatrix} 18 & 1 & -4 \\ 9 & -1 & -9 \\ 4 & 16 & 11 \end{bmatrix}$

(b) $A.B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 8 \\ 2 & -2 & 5 \end{bmatrix} \begin{bmatrix} -16 & 1 & 8 \\ -11 & 1 & 5 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -16+11+6 & 1-1 & 8-5-3 \\ -16+16 & 1 & 8-8 \\ -32+22+10 & 2-2 & 16-10-5 \end{bmatrix}$

$\Rightarrow$ $A.B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$. So, by definition $B = A^{-1}$.

(c) Since A is invertible then by Invertible Matrix Theorem, the columns of A span R^3.

4. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined as

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x - y - z \\ 2x + y - w \end{bmatrix}.$$

Find the matrix of the linear transformation T. Determine if the linear transformation T is one-to-one.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $T(e_4) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

So $T(x) = Ax$, where $A = [T(e_1) \ T(e_2) \ T(e_3) \ T(e_4)]$

$$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

Consider the homogeneous equation $Ax = 0$, having the augmented matrix

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 3 & 2 & -1 & 0 \end{bmatrix}$$

↑ ↑
non-pivot columns are 3rd and 4th columns.
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x_3, x_4: free variables

It has nontrivial solutions, then the columns are linearly dependent.

So, $T(x) = Ax$ is not one-to-one transformation.