KEY

1. (a) Find an equation to describe the relation between a, b, c in order to the following system be consistent

12pts

$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= a \\ 2x_2 - 4x_3 &= b \\ -4x_1 + 6x_2 - 8x_3 &= c \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 5 & a \\ 0 & 2 & -4 & b \\ -4 & 6 & -8 & c \end{bmatrix} \xrightarrow{4R_1+R_3 \to R_3} \begin{bmatrix} 1 & -3 & 5 & a \\ 0 & 2 & -4 & b \\ 0 & -6 & +12 & 4a+c \end{bmatrix} \xrightarrow{3R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & -3 & 5 & a \\ 0 & 2 & -4 & b \\ 0 & 0 & 0 & 4a+3b+c \end{bmatrix}$$

System is consistent if $4a+3b+c=0$

(b) For given

13pts

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 7 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

determine whether **b** can be written as linear combination of the $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ such that $A = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b \Rightarrow x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 7 & -4 & -3 \end{bmatrix} \xrightarrow{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & -1 & 0 & 3 \end{bmatrix} \xrightarrow{-R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 3 & 5 & -7 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3 \to R_3} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 5 & 2 \end{bmatrix} \Rightarrow \begin{aligned} &x_1 - 4x_2 + 2x_3 = 3 \Rightarrow x_1 + 12 + \tfrac{4}{5} = 3 \\ &x_2 = -3 \qquad\qquad x_1 = -9 - \tfrac{4}{5} \\ &5x_3 = 2 \to x_3 = \tfrac{2}{5} \qquad x_1 = -\tfrac{49}{5} \end{aligned}$$

$b = -\frac{49}{5} v_1 - 3 v_2 + \frac{2}{5} v_3$, b can be written as linear combination of v_1, v_2 and v_3.

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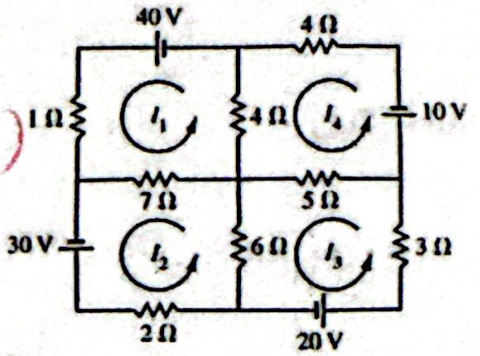
2. **(a)** Write a matrix equation that determines the loop currents. Do not solve!

10pts



$12I_1 - 7I_2 - 4I_4 = 40$

$-7I_1 + 15I_2 - 6I_3 = 30$

$-6I_2 + 14I_3 - 5I_4 = 20$

$-4I_1 - 5I_3 + 13I_4 = -10$

$$\begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$$

5

(b) Consider the linear transformation T whose standard matrix is

15pts

$$A = \begin{bmatrix} -5 & 10 & -5 & 5 \\ 8 & 4 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}.$$

(i) Is T 1-1? Show your work

(ii) Is T onto? Show your work.

$7 + \frac{3^2}{5}$

$$\begin{bmatrix} -5 & 10 & -5 & 5 \\ 8 & 4 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_1 \to R_1} \sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 8 & 4 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix} \begin{matrix} -8R_1+R_2 \to R_2 \\ -4R_1+R_3 \to R_3 \\ \sim \\ 3R_1+R_4 \to R_4 \end{matrix} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 20 & -12 & 15 \\ 0 & -1 & 1 & 1 \\ 0 & -8 & 8 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 8 & 35 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

(i) $Ax = 0$ has only the trivial solution considering the echelon form of A. So T is 1-1 3

(ii) $Ax = b$ has at least one solution, T is onto.

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3. **(a)** Find the inverse of $A = \begin{bmatrix} 4 & -2 & 5 \\ 1 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$ **using row operations, if it exists. State whether the column vectors of the given matrix linearly independent or not.**

13pts

$$\left[\begin{array}{ccc|ccc} 4 & -2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 1 & 0 & 0 \\ -1 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -6 & 5 & 1 & -4 & 0 \\ 0 & -2 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & -1/2 \\ 0 & 0 & -1 & 1 & -7 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 13/2 & 5/2 \\ 0 & 0 & 1 & -1 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -11/2 & -5/2 \\ 0 & 1 & 0 & -1 & 13/2 & 5/2 \\ 0 & 0 & 1 & -1 & 7 & 3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -11/2 & -5/2 \\ -1 & 13/2 & 5/2 \\ -1 & 7 & 13 \end{bmatrix}$$

Since A is row equavellent to identity matrix colomns of A are linearly independent

(b) Find an LU factorization of the given matrix

12pts

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \overset{\frac{1}{3}R_1 \to R_1}{\sim} \begin{bmatrix} 1 & -2 & 1 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \underset{R_1+R_3 \to R_3}{\overset{-6R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -4 \\ 0 & 5 & 1 \end{bmatrix} \overset{-R_2+R_3 \to R_3}{\sim}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & \boxed{5} \end{bmatrix} \qquad U = \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}$$

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4. **(a)** Compute the determinant of the given matrix using cofactor expansion.

10pts

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -5 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

Cofactor expansion across the first row:

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = C_{11} + C_{12}$, $C_{13} = 0$ (2)

$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} = -4$ (3) $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} -5 & 4 \\ 0 & -1 \end{vmatrix} = -5$ (3)

$\det A = -4 - 5 = -9$ (2)

15pts **(b)** Use Cramer's rule to compute the solutions of the following system.

$$\begin{aligned} x_1 + x_2 &= 2 \\ -5x_1 \qquad + 4x_3 &= 0 \\ x_2 - x_3 &= -1 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -5 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}_{b}$$

A is the same matrix in part (a), $|A| = -9$

$x_1 = \frac{|A_1(b)|}{|A|} = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 4 \\ -1 & 1 & -1 \end{vmatrix}}{-9} = \frac{-4\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}}{-9} = \frac{-12}{-9} = \frac{4}{3}$ (5)

$x_2 = \frac{|A_2(b)|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 0 \\ -5 & 0 & 4 \\ 0 & -1 & -1 \end{vmatrix}}{-9} = \frac{4-10}{-9} = \frac{-6}{-9} = \frac{2}{3}$ (5)

$x_3 = \frac{|A_3(b)|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 2 \\ -5 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}}{-9} = \frac{-(-5)\cdot\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}{-9} = \frac{5\cdot(-3)}{-9} = \frac{5}{3}$ (5)

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