Matrix Vector multipl

$$A_{m\times n}\, X_{n\times 1} = \begin{bmatrix} R_1 X \\ R_2 X \\ \vdots \\ R_m X \end{bmatrix}_{m\times 1}$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2.1 + 2.3 - 1.1$$
$$= 7$$

1. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

Matrix Vector multiplication

$$A_{m\times n}\, X_{n\times 1} = \begin{bmatrix} R_1 x \\ R_2 x \\ \vdots \\ R_m x \end{bmatrix}_{m\times 1}$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 2.1 + 2.3 - 1.1$$
$$= 7$$

① $\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1.2 + 2.1 \\ -1.2 + 3.1 \\ 2.2 + 1.1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}_{3\times 1}$

② $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ undefined

$3\times 4 \quad 3\times 1$

$3\times 4 \quad 4\times 1$

Def 1: Consider a linear system

$$A_{m\times n}\, x = b$$

Then the augmented matrix of the system is another matrix of the size $m\times(n+1)$ given by $[A|b]$

for example augmented matrix of system (1) is

$$\left[\begin{array}{ccccc|c} 2 & -1 & 3 & 1 & -1 & 1 \\ -1 & 2 & -1 & -1 & 2 & 0 \\ 1 & 3 & -2 & 0 & 1 & 2 \end{array}\right] (2)$$

$(1) \equiv (2)$

f.e. write the system corresponding to

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 7 \end{array}\right] \end{array}$$

There are 3 rows $\Rightarrow$ 3 eqns

There must be 3 unknowns say x, y, z

3rd row: $2z$

2nd row: $2y + 3z = 4$

1st row: $2x - z = 1$

en orm of the sys.

Elementary row operations

① Scaling: We can multiply any row with a constant

② Interchanging: We can interchange any two rows which we want

③ Replacement: We can multiply a row with a constant and add it to another row.

They don't change the sol set.

f.e. Write the augmented matrix for the system and apply row operations to generate a simple equivalent system

$2x - y + z - w = 1$

$x - y + z - 2w = 2$

$-x + 2y - 2z + 3w = 6$

i.e. Write the system corresponding to

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 7 \end{array}\right]$$

There are 3 rows $\Rightarrow$ 3 eqns

There must be 3 unknowns say x, y, z.

3rd row: 2 [illegible]

2nd row: $2y + 3z = 4$

1st row: $2x - z = 1$

open form of the sys.

Augmented matrix $\longrightarrow$ Echelon form.

Gauss Elimination.

<u>Solution Cases:</u>

Examples

(1)

$$x - 2y + z = 1$$
$$-2x + y - 2z = 4$$
$$x - 5y + z = 2$$

Augmented matrix

$\downarrow$

Row op.

$\downarrow$

Echelon

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ -2 & 1 & -2 & 4 \\ 1 & -5 & 1 & 2 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -3 & 0 & 6 \\ 0 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -3 & 0 & 6 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

3rd row: $0 \neq -5$

No sol.

Sys. is inconsistent!

1st row: $x - 2y - z = 1$

$x + 3 - \frac{1}{8} = 1$

$x = \frac{1}{8} - 2 = -\frac{15}{8}$

Sol. set $= \left\{ \begin{bmatrix} -15/8 \\ -3/2 \\ 1/8 \end{bmatrix} \right\}$

Ex2. part 2: Solve

$x + y + z = 1$

$x - y + z = 0$

$3x + y + 3z = 2$

$x + y - z = 5$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 1 & 1 & -1 & 5 \end{array}\right] \xrightarrow[\substack{-3R_1+R_3 \to R_3 \\ -R_1+R_4 \to R_4}]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

Unique sol!

$-2z = 4$

$z = -2$

$-2y = -1 \rightarrow y = 1/2$

$x + y + z = 1$

$x - \frac{3}{2} = 1 \rightarrow x = 5/2$

Sol set $= \left\{ \begin{bmatrix} 5/2 \\ 1/2 \\ -2 \end{bmatrix} \right\}$

$\xrightarrow[-3R_1+R_3 \to R_3]{R_1+R_2 \to R_2}$

$\xrightarrow{5R_2+R_3 \to R_3}$

3rd row:

2nd row:

$y = -1 \rightarrow y = 1/2$

$+ y + z = 1$

$x - \frac{3}{2} = 1 \rightarrow x = 5/2$

Sol set $= \left\{ \begin{bmatrix} 5/2 \\ 1/2 \\ -2 \end{bmatrix} \right\}$

Ex 3: Infinitely many sol^ns.

$$A_{m \times n} x = b$$

① Suppose that $m \geq n$

If the number of leading entries is less than number of unknowns and if there is one entirely zero row, then we have inf many sol^ns.

part 1:

$y + z = 1$
$2y - z = 2$
$-4y + 3z = 0$

$$\left[\begin{array}{c|c} 1 & 1 \\ -1 & 2 \\ 3 & 0 \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ 1 & -2 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ inf. many solns

free parameter $z = t$ $(t \in \mathbb{R})$.

2^{nd} row $2y - z = 2$

$y = 1 + t/2$

1^{st} row $x - y + z = 1$

$x - 1 - \frac{t}{2} + t = 1$

$x = 2 - \frac{t}{2}$

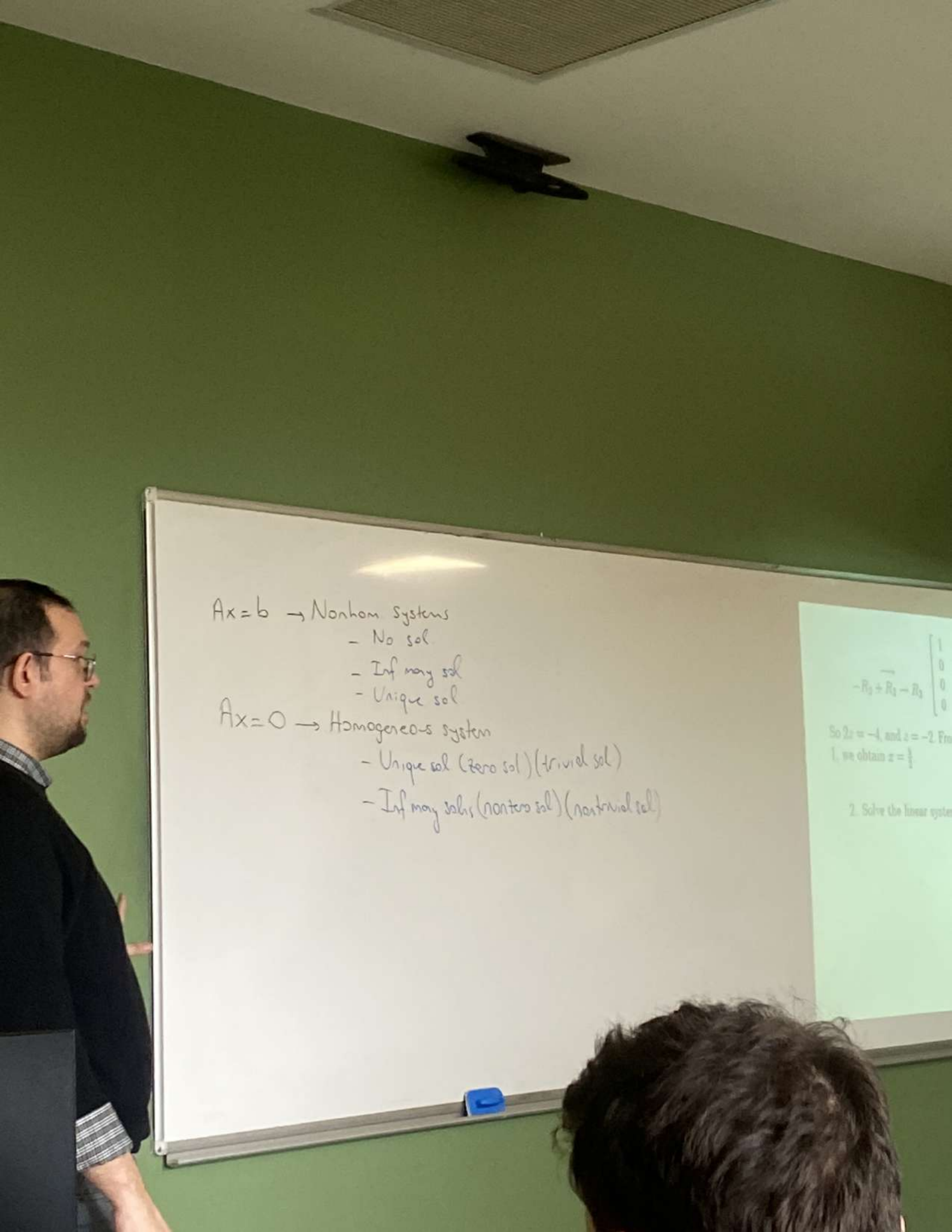
Sol. set $= \left\{ \begin{bmatrix} 2 - t/2 \\ t/2 + 1 \\ t \end{bmatrix}, t \in \mathbb{R} \right\}$

Ex 3: Infinitely many solns.

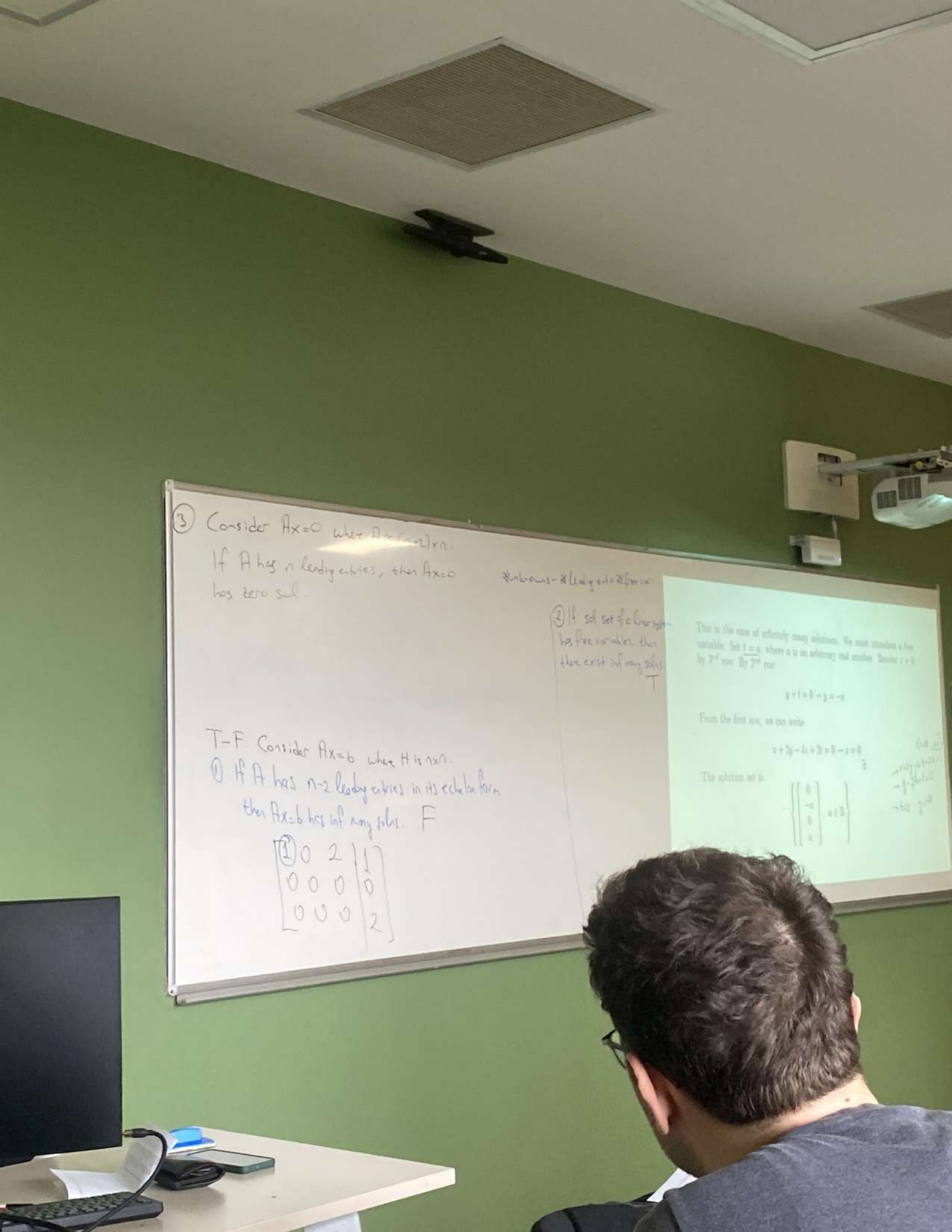
$A_{m \times n} x = b$

1. Suppose that $m \geq n$

If the number of leading entries is less than number of unknowns and if there is one entirely zero row, then we have inf many solns.



$Ax = b \rightarrow$ Nonhom systems
- No sol
- Inf many sol
- Unique sol
$Ax = 0 \rightarrow$ Homogeneous system
- Unique sol (zero sol) (trivial sol)
- Inf many sol's (nonzero sol) (nontrivial sol)

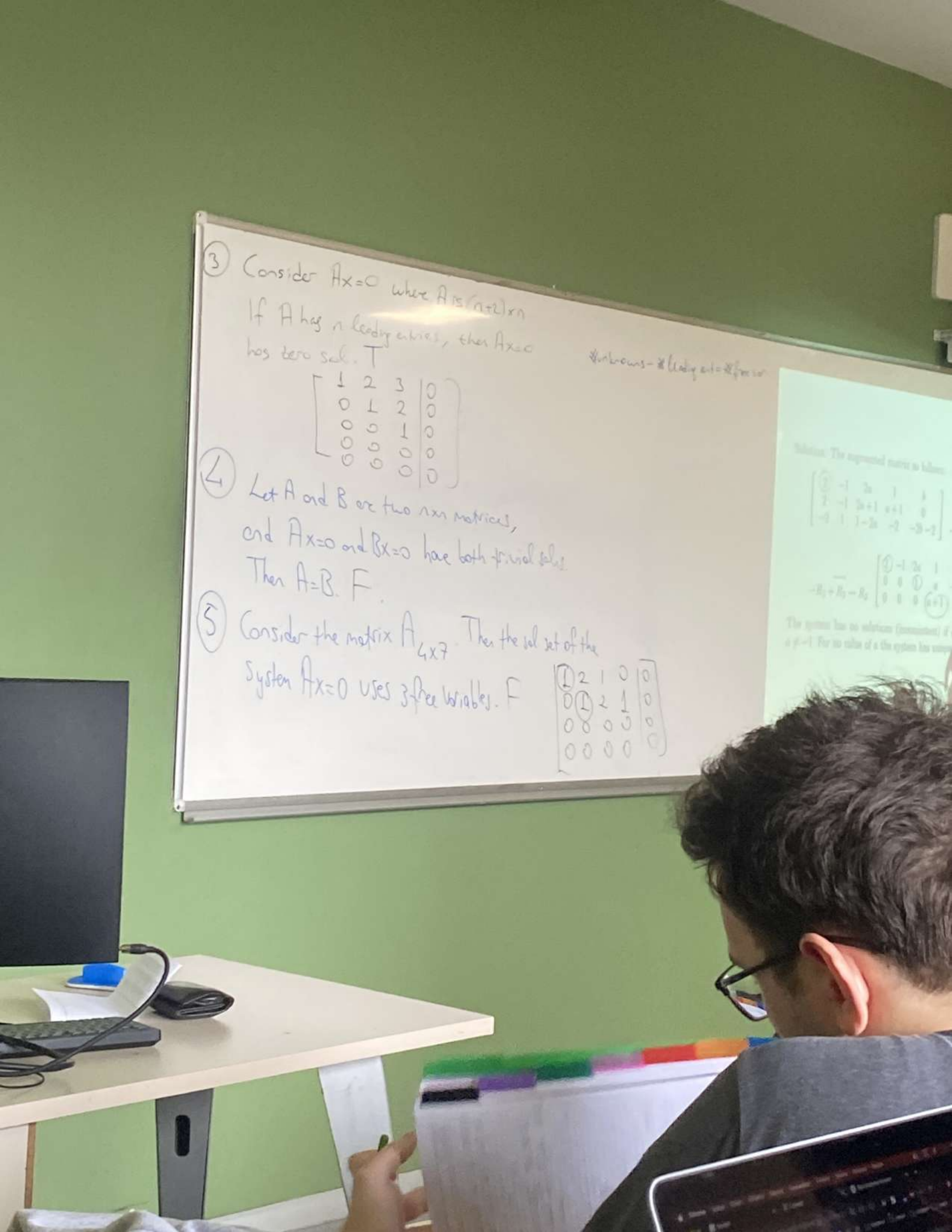

③ Consider Ax=0 where A is (n-2)×n.
If A has n leading entries, then Ax=0
has zero sol.
#unknowns − #leading ent = #free var
② If sol set of a linear system has free variables, then there exist inf many solns.
T
T-F Consider Ax=b where A is n×n.
① If A has n-2 leading entries in its echelon form then Ax=b has inf many solns.
F
This is the case of infinitely many solutions.
From the first row, we can write
The solution set is

(3) Consider $Ax=0$ where A is $(n+2)\times n$.
If A has n [illegible], then $Ax=0$ has zero sol.. T

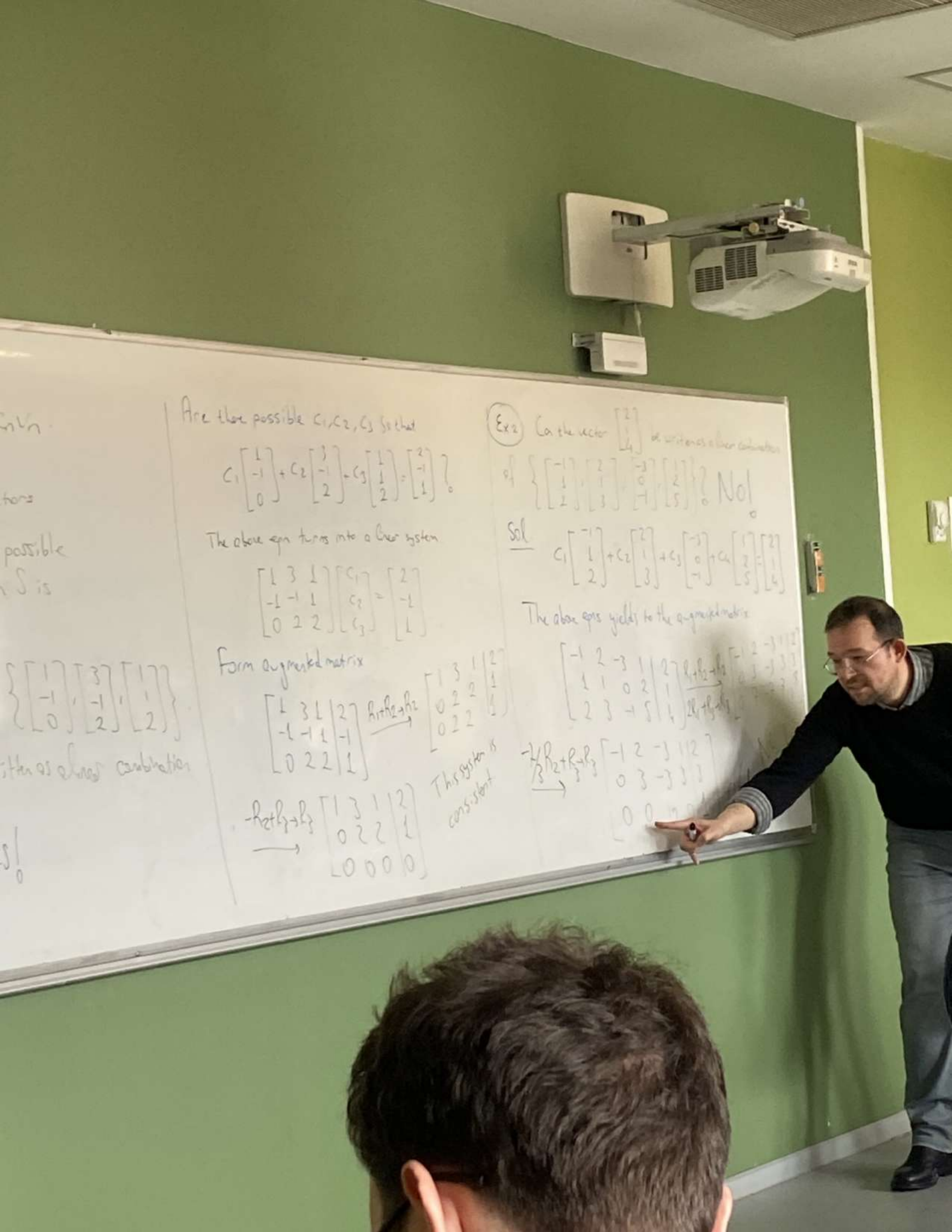
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

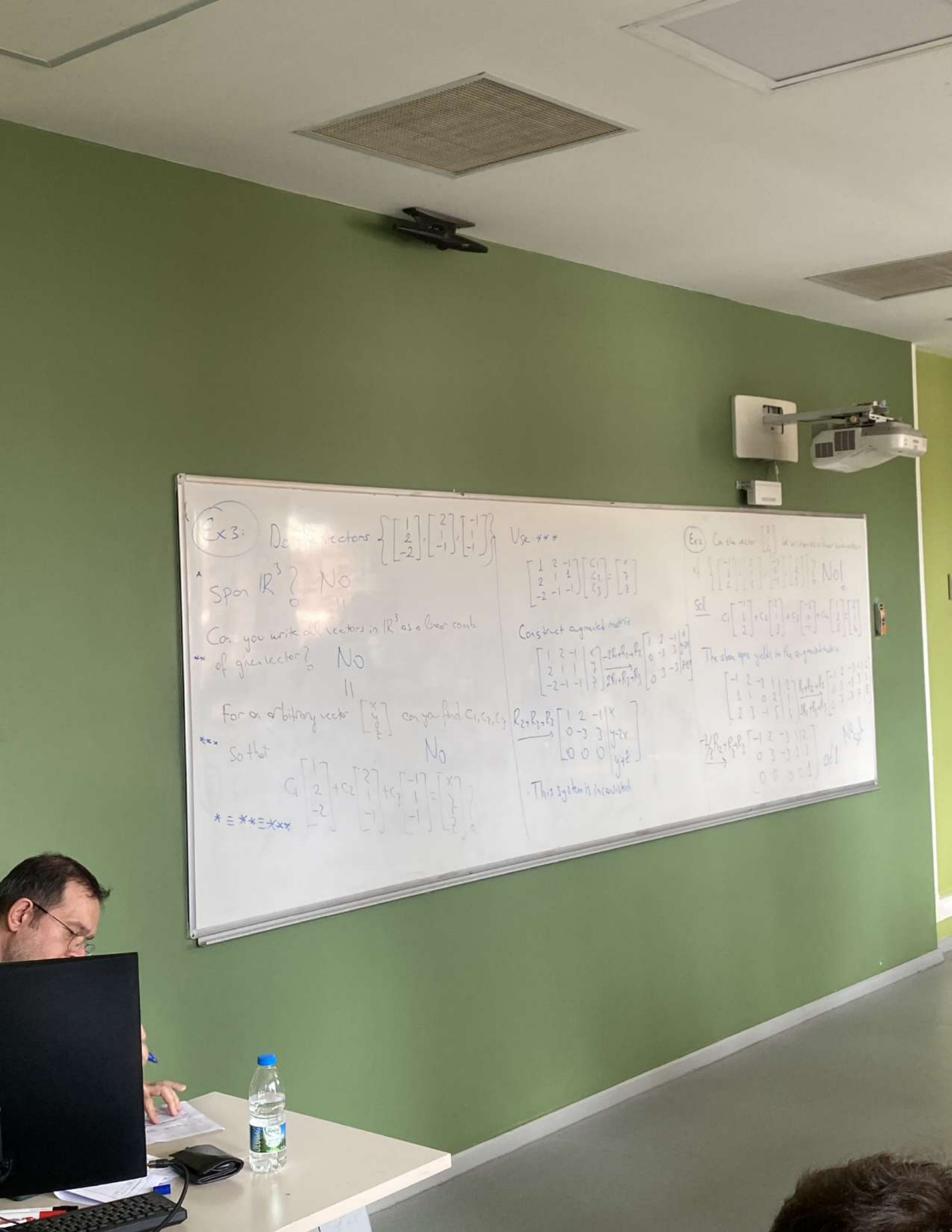
(4) Let A and B are two $n\times n$ matrices,
and $Ax=0$ and $Bx=0$ have both trivial sol's.
Then $A=B$. F

(5) Consider the matrix $A_{4\times 7}$. Then the sol set of the
System $Ax=0$ uses 3 free variables.



3 Consider Ax=0 where A is (n+2)xn
If A has n leading entries, then Ax=0 has zero sol. T
#unknowns - # leading ent = # free var
1 2 3 | 0
0 1 2 | 0
0 0 1 | 0
0 0 0 | 0
0 0 0 | 0
4 Let A and B are two nxn matrices,
and Ax=0 and Bx=0 have both trivial sol.
Then A=B. F
5 Consider the matrix A 4x7. Then the sol set of the
system Ax=0 uses 3 free variables. F
1 2 1 0 | 0
0 1 2 1 | 0
0 0 0 0 | 0
0 0 0 0 | 0


Are there possible c1, c2, c3 so that
c1 [1, -1, 0] + c2 [3, -1, 2] + c3 [1, 1, 2] = [2, -1, 1] ?
The above eqn turns into a linear system
[1 3 1; -1 -1 1; 0 2 2] [c1; c2; c3] = [2; -1; 1]
Form augmented matrix
[1 3 1 | 2; -1 -1 1 | -1; 0 2 2 | 1] R1+R2→R2 [1 3 1 | 2; 0 2 2 | 1; 0 2 2 | 1]
-R2+R3→R3 [1 3 1 | 2; 0 2 2 | 1; 0 0 0 | 0]
This system is consistent
Ex2 Can the vector [2, 1, 4] be written as a linear combination
of {[-1, 1, 2], [2, 1, 3], [-3, 0, -1], [1, 2, 5]} ? No!
Sol c1 [-1, 1, 2] + c2 [2, 1, 3] + c3 [-3, 0, -1] + c4 [1, 2, 5] = [2, 1, 4]
The above eqns yields to the augmented matrix
[-1 2 -3 1 | 2; 1 1 0 2 | 1; 2 3 -1 5 | 4] R1+R2→R2, 2R1+R3→R3
-7/3 R2+R3→R3 [-1 2 -3 1 | 2; 0 3 -3 3 | 3; 0 0 …]
possible
S is
written as a linear combination



Ex 3:
Span $\mathbb{R}^3$? No
Can you write all vectors in $\mathbb{R}^3$ as a linear comb of given vector? No
For an arbitrary vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can you find c_1, c_2, c_3
So that
No
$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$?
* ≡ ** ≡ ***
Use ***
$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Construct augmented matrix
$R_2 + R_3 \to R_3$
$\left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 0 & -3 & 3 & y-2x \\ 0 & 0 & 0 & y+z \end{array}\right]$
This system is inconsistent
Ex
No!
Sol

$$\begin{bmatrix} & 2 & -1 \\ & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

...ruct augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 2 & 1 & 1 & y \\ -2 & -1 & -1 & z \end{array}\right] \xrightarrow[2R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 0 & -3 & 3 & y-2x \\ 0 & 3 & -3 & z+2x \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 0 & -3 & 3 & y-2x \\ 0 & 0 & 0 & y+z \end{array}\right]$$

System is inconsistent

T-F

① In order to span $\mathbb{R}^n$, we exactly need n vectors from $\mathbb{R}^n$ F

Corrected: In order to span $\mathbb{R}^n$, we need at least n vectors from $\mathbb{R}^n$

② Suppose that the square matrix $A_{n\times n}$ has n leading entries. Then, its columns span $\mathbb{R}^n$ T

③ Suppose that the columns of the matrix A span $\mathbb{R}^n$. Then $Ax=b$ has unique sol F

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$, v_2, v_3 \}$

Use ***

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Construct augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 2 & 1 & 1 & y \\ -2 & -1 & -1 & z \end{array}\right] \xrightarrow[2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 0 & -3 & 3 & y-2x \\ 0 & 3 & -3 & z+2x \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & x \\ 0 & -3 & 3 & y-2x \\ 0 & 0 & 0 & y+z \end{array}\right]$$

This system is inconsistent

T-F

① In order to span $\mathbb{R}^n$, we exactly need n vectors from $\mathbb{R}^n$ F

Corrected: In order to span $\mathbb{R}^n$, we need at least n vectors from $\mathbb{R}^n$

② Suppose that the square matrix $A_{n\times n}$ has n leading entries. Then, its columns span $\mathbb{R}^n$. T

③ Suppose that the columns of the matrix A span $\mathbb{R}^n$. Then $Ax=b$ has unique sol F $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

④ Suppose that $\vec{u} \in \text{span}\{v_1, v_2, v_3\}$

Then $\text{span}\{u, v_1, v_2, v_3\} \neq \text{span}\{v_1, v_2, v_3\}$

Use

Co

$R_2 +$

(4) Suppose that $\vec{u} \in \text{span}\{v_1, v_2, v_3\}$

Then $\text{span}\{u, v_1, v_2, v_3\} \neq \text{span}\{v_1, v_2, v_3\}$ F

Corrected $\text{span}\{u, v_1, v_2, v_3\} = \text{span}\{v_1, v_2, v_3\}$

$$c_1 u + c_2 v_1 + c_3 v_2 + c_4 v_3$$

$$u = d_1 v_1 + d_2 v_2 + d_3 v_3$$

$$c_1(d_1 v_1 + d_2 v_2 + d_3 v_3) + c_2 v_1 + c_3 v_2 + c_4 v_3$$

$$(c_1 d_1 + c_2) v_1 + (c_1 d_2 + c_3) v_2 + (c_1 d_3 + c_4) v_3$$

④ Suppose that $\vec{u} \in \text{span}\{v_1, v_2, v_3\}$

Then $\text{span}\{u, v_1, v_2, v_3\} \neq \text{span}\{v_1, v_2, v_3\}$

F

Corrected: $\text{span}\{u, v_1, v_2, v_3\} = \text{span}\{v_1, v_2, v_3\}$

$$c_1u + c_2v_1 + c_3v_2 + c_4v_3$$

$$u = d_1v_1 + d_2v_2 + d_3v_3$$

$$c_1(d_1v_1 + d_2v_2 + d_3v_3) + c_2v_1 + c_3v_2 + c_4v_3$$

$$(c_1d_1 + c_2)v_1 + (c_1d_2 + c_3)v_2 + (c_1d_3 + c_4)v_3$$

⑤ Let $A_{n\times n}$ be a square matrix and its known that $Ax=0$ has nontrivial sol. Then columns of A dont span $\mathbb{R}^n$.

T

$A_{n\times n}x = 0 \xrightarrow{\text{nonzero sol}}$ Inf many sol case,

$\downarrow$

leading entries $< n$

$\downarrow$

dont span $\mathbb{R}^n$

$Ax = b$ consistent

if $b \in \text{span}\{a_1, a_2, \dots, a_n\}$

PROPERTIES OF THE MATRIX-VECTOR PRODUCT A

Theorem 5: If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

a. $A(u+v) = Au + Av$;

b. $A(cu) = c(Au)$.

Proof: For simplicity, take $n = 3$, $A = [a_1 \ a_2 \ a_3]$ and u, v in $\mathbb{R}^3$.

For $i = 1, 2, 3$, let u_i and v_i be the ith entries in u and v, respectively.