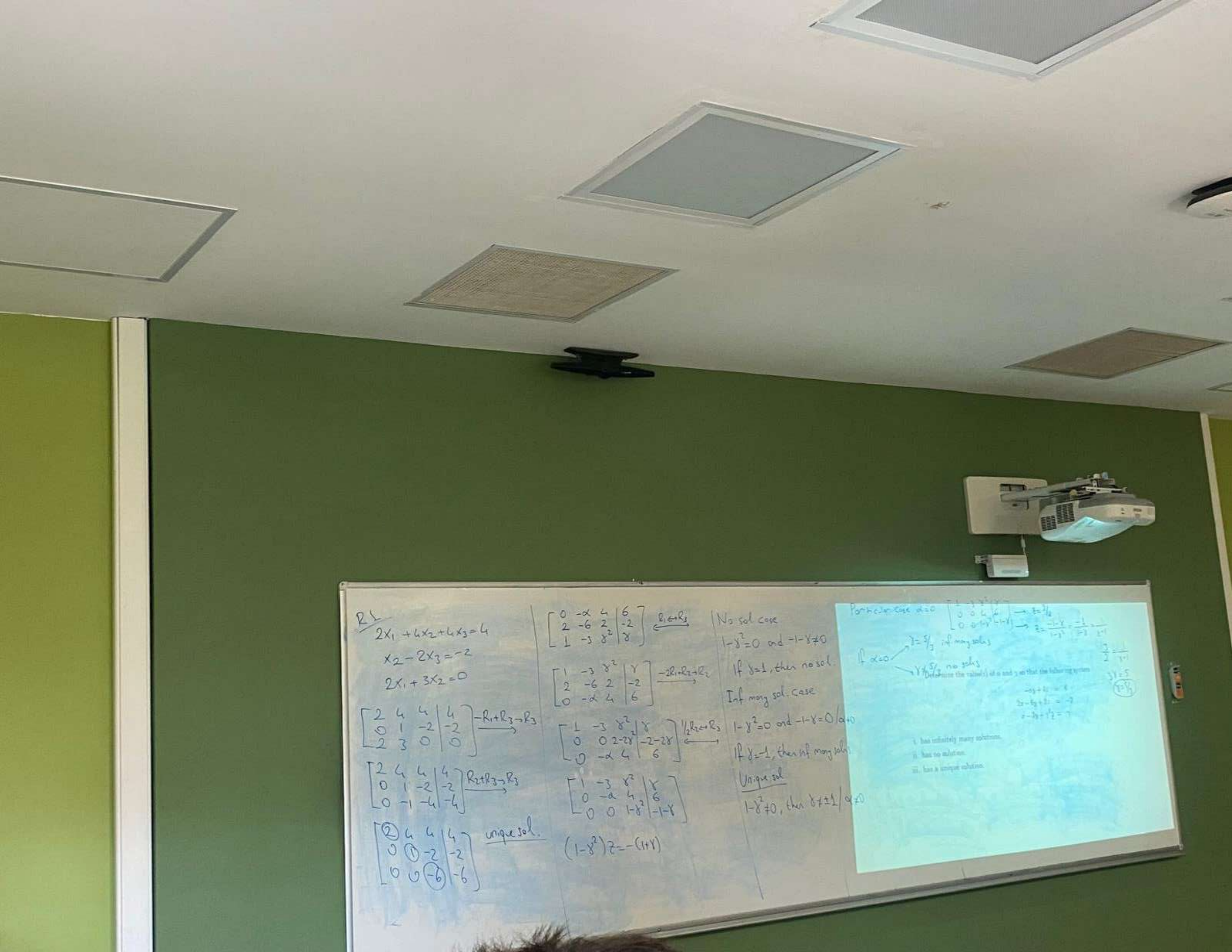
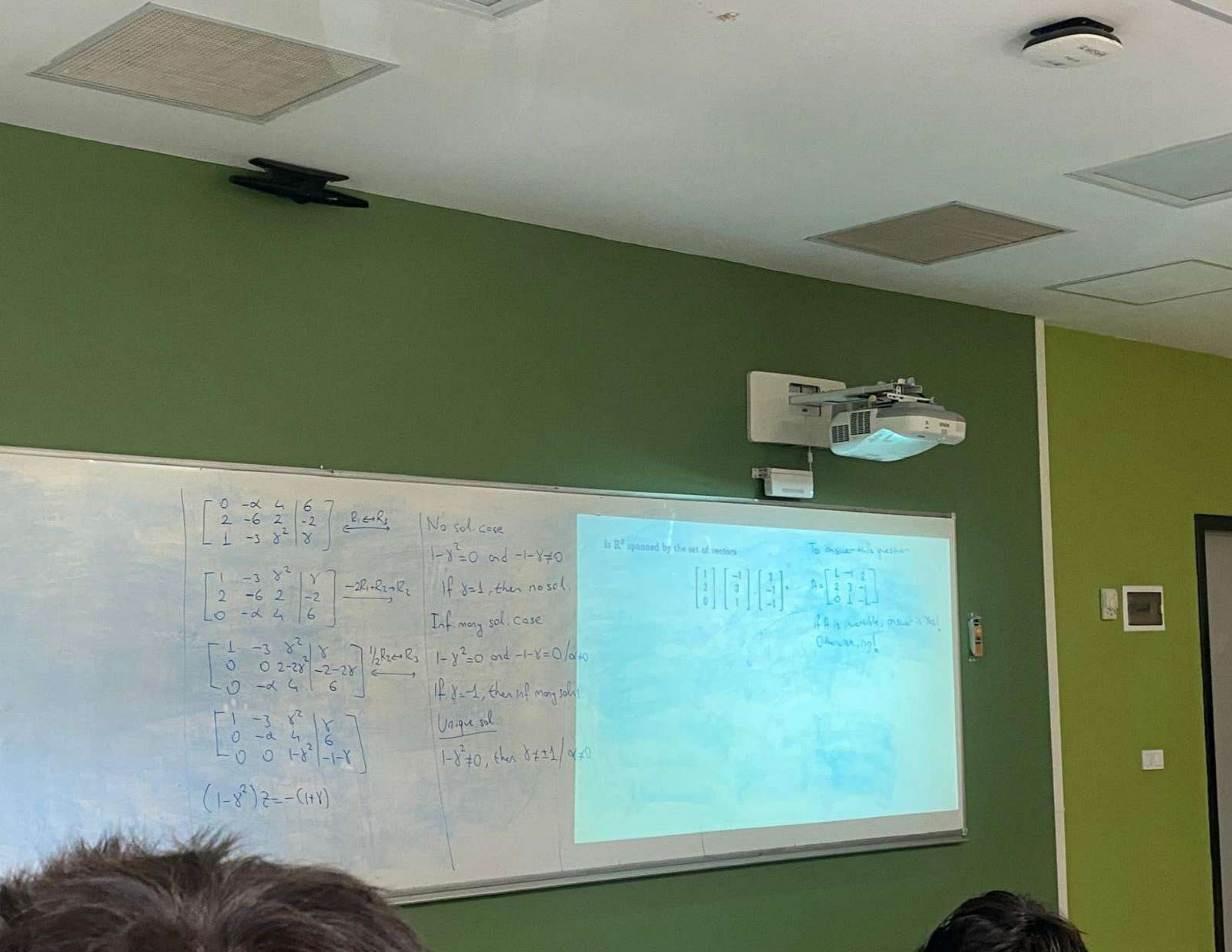
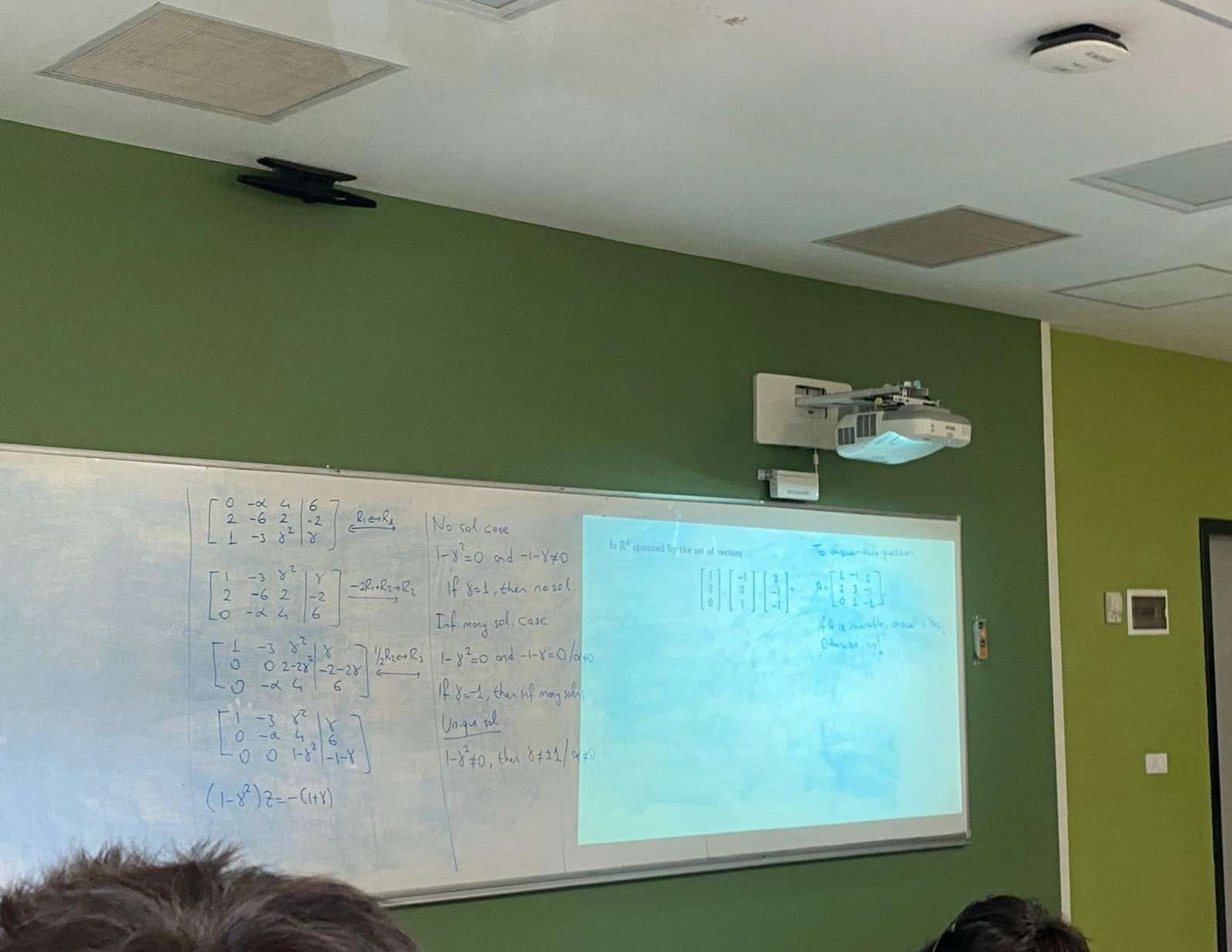

Ex
2x1 + 4x2 + 4x3 = 4
x2 − 2x3 = −2
2x1 + 3x2 = 0
−R1+R3→R3
R2+R3→R3
unique sol.
R1↔R3
−2R1+R2→R2
1/2R2+R3
(1−γ²)z = −(1+γ)
No sol case
1−γ²=0 and −1−γ≠0
If γ=1, then no sol.
Inf many sol. case
1−γ²=0 and −1−γ=0 / α≠0
If γ=−1, then inf many sols
Unique sol
1−γ²≠0, then γ≠±1 / α≠0
Particular case α=0
If α=0
no sols
i. has infinitely many solutions.
ii. has no solution.
iii. has a unique solution.



$\left[\begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_3}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xleftrightarrow{1/2 R_2 \leftrightarrow R_3}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{array}\right]$
$(1-\gamma^2)z = -(1+\gamma)$
No sol. case
$1-\gamma^2=0$ and $-1-\gamma \neq 0$
If $\gamma=1$, then no sol.
Inf many sol. case
$1-\gamma^2=0$ and $-1-\gamma=0$ / $\alpha \neq 0$
If $\gamma=-1$, then inf many sols.
Unique sol
$1-\gamma^2 \neq 0$, then $\gamma \neq \pm 1$ / $\alpha \neq 0$
Is $\mathbb{R}^3$ spanned by the set of vectors
$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$?
To answer this question:
$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
If A is invertible, answer is Yes!
Otherwise, no!



$\left[\begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_3}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xleftrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$
$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{array}\right]$
$(1-\gamma^2)z = -(1+\gamma)$
No sol case
$1-\gamma^2=0$ and $-1-\gamma \neq 0$
If $\gamma=1$, then no sol.
Inf many sol. case
$1-\gamma^2=0$ and $-1-\gamma=0$ / $\alpha \neq 0$
If $\gamma=-1$, then inf many sol.
Unique sol
$1-\gamma^2 \neq 0$, then $\gamma \neq \pm 1$ / $\alpha \neq 0$
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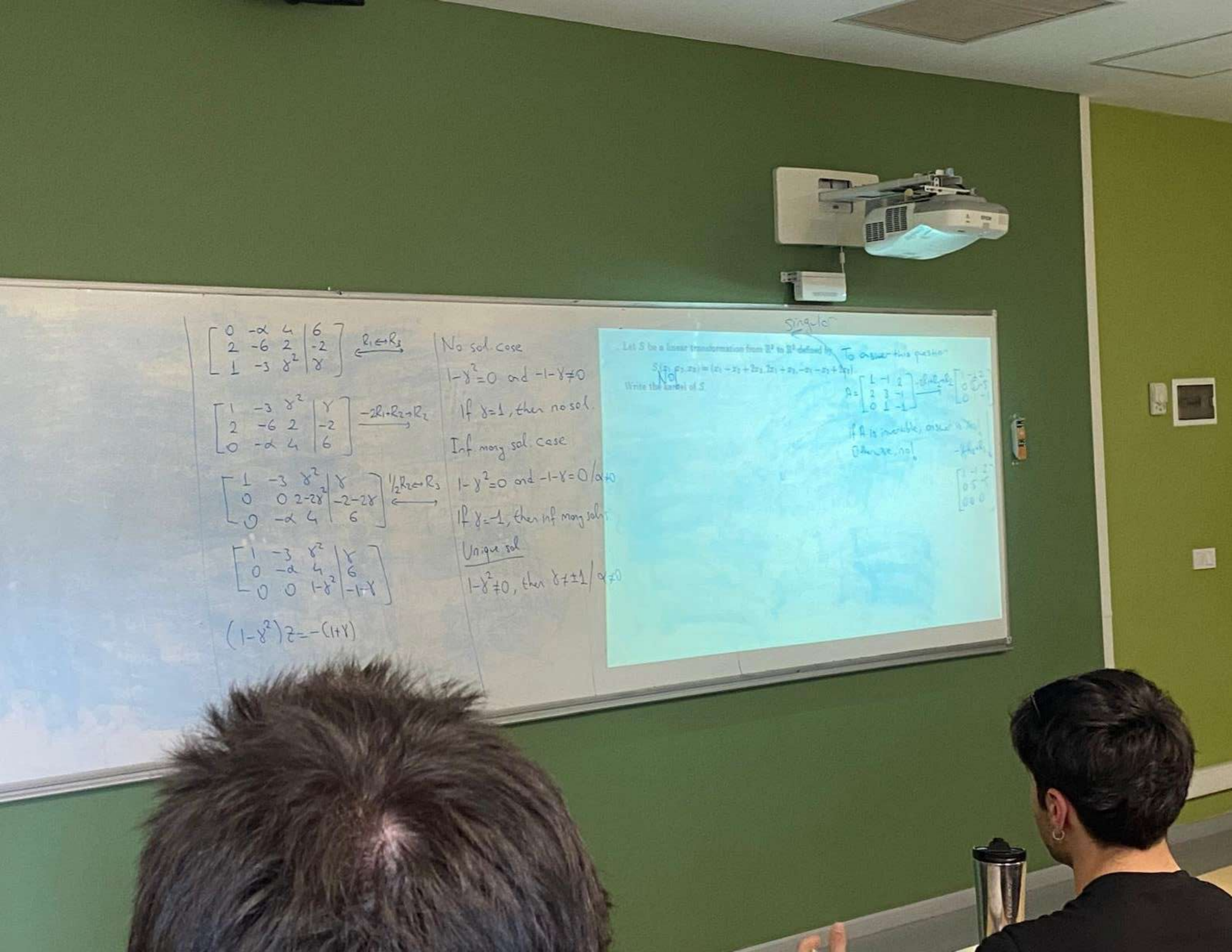
$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$?
To answer this question
$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
Otherwise, no!



R1↔R3
-2R1+R2→R2
1/2R2↔R3
No sol. case
1-γ²=0 and -1-γ≠0
If γ=1, then no sol.
Inf many sol. case
1-γ²=0 and -1-γ=0 / α≠0
If γ=-1, then inf many sol.
Unique sol
1-γ²≠0, then γ≠±1 / α≠0
(1-γ²)z=-(1+γ)
singular
To answer this question
Let S be a linear transformation from
Write the kernel of S.
No

$\xleftrightarrow{R_1 \leftrightarrow R_3}$

$\xrightarrow{-2R_1+R_2 \to R_2}$

$\xleftrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$

No sol. case

$1-\gamma^2=0$ and $-1-\gamma \neq 0$

If $\gamma=1$, then no sol.

Inf. many sol. case

$1-\gamma^2=0$ and $-1-\gamma=0$ / $\alpha \neq 0$

If $\gamma=-1$, then inf many sol'ns

<u>Unique sol</u>

$1-\gamma^2 \neq 0$, then $\gamma \neq \pm 1$ / $\alpha \neq 0$

Let S be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$S(x_1,x_2,x_3)=(x_1-x_2+2x_3,\ 2x_1+x_2,\ -x_1-x_2+2x_3)$$

Write the kernel of S.

Singular

$$x_1-x_2+2x_3=0$$
$$2x_1+x_2=0$$
$$-x_1-x_2+2x_3=0$$

We need to solve $Ax=0$ if

$$A=\begin{bmatrix}1&-1&2\\2&1&0\\-1&-1&2\end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{ccc|c}1&-1&2&0\\2&1&0&0\\-1&-1&2&0\end{array}\right]\xrightarrow[R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1&-1&2&0\\0&3&-4&0\\0&-2&4&0\end{array}\right]$$

$$\xrightarrow{\frac{2}{3}R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-1&2&0\\0&3&-4&0\\0&0&\frac{4}{3}&0\end{array}\right]$$

$\mathrm{Ker}\, S=\{\vec{0}\}$

S is 1-1 and onto

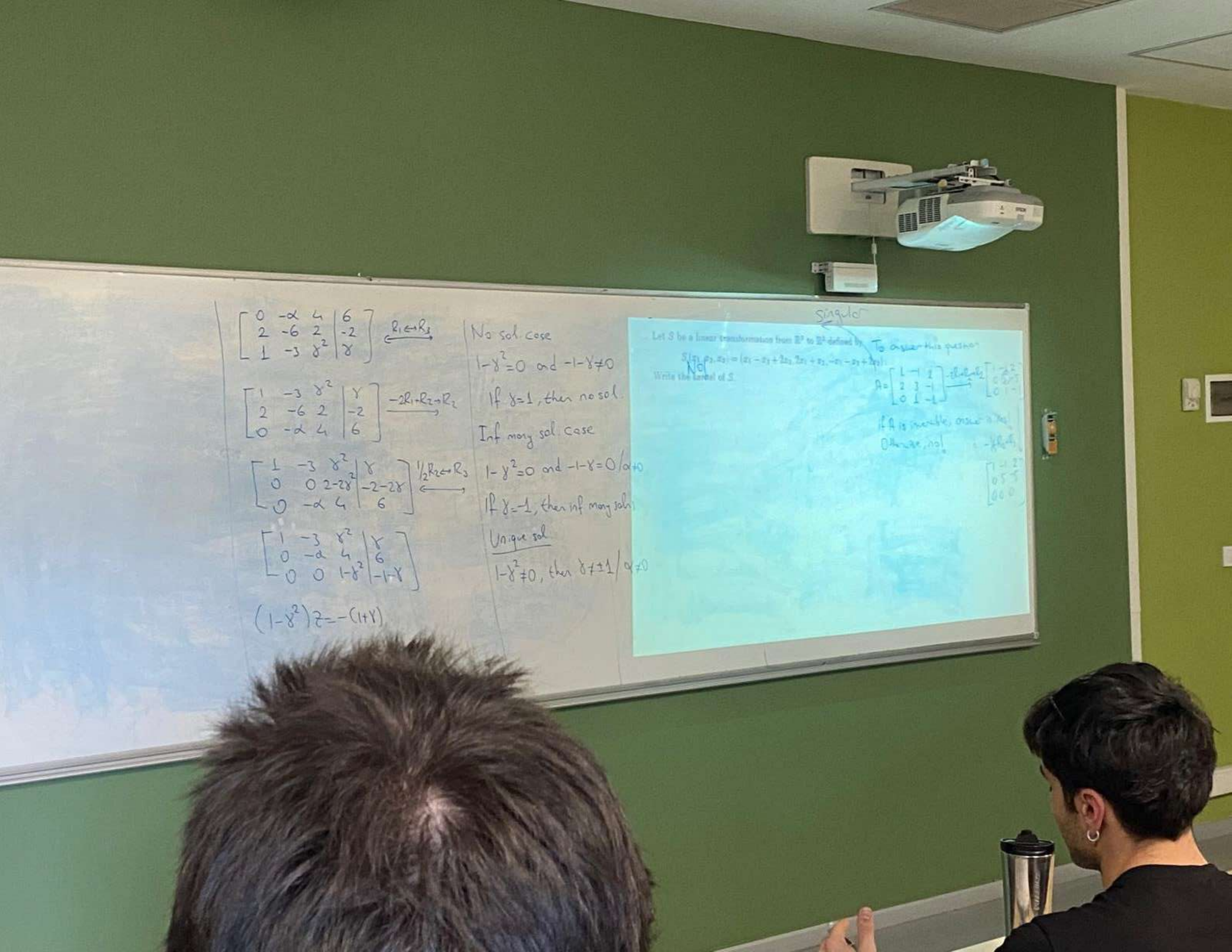
To answer this question

$$A=\begin{bmatrix}1&-1&2\\2&3&-1\\0&1&-1\end{bmatrix}\xrightarrow{-2R_1+R_2\to R_2}\begin{bmatrix}1&-1&2\\0&5&-5\\0&1&-1\end{bmatrix}$$

If A is invertible, answer is Yes!

Otherwise, no!

$$\xrightarrow{-\frac{1}{5}R_2+R_3}\begin{bmatrix}1&-1&2\\0&5&-5\\0&0&0\end{bmatrix}$$



R1↔R3
−2R1+R2→R2
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(1−γ²)z=−(1+γ)
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1−γ²=0 and −1−γ=0 / α≠0
If γ=−1, then inf many sols
Unique sol
1−γ²≠0, then γ≠±1 / α≠0
Singular
Let S be a linear transformation from ℝ³ to ℝ³ defined by
Write the kernel of S.
To answer this question
No!
If A is invertible, answer is Yes!
Otherwise, no!

Check the invertibility of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A has no inverse

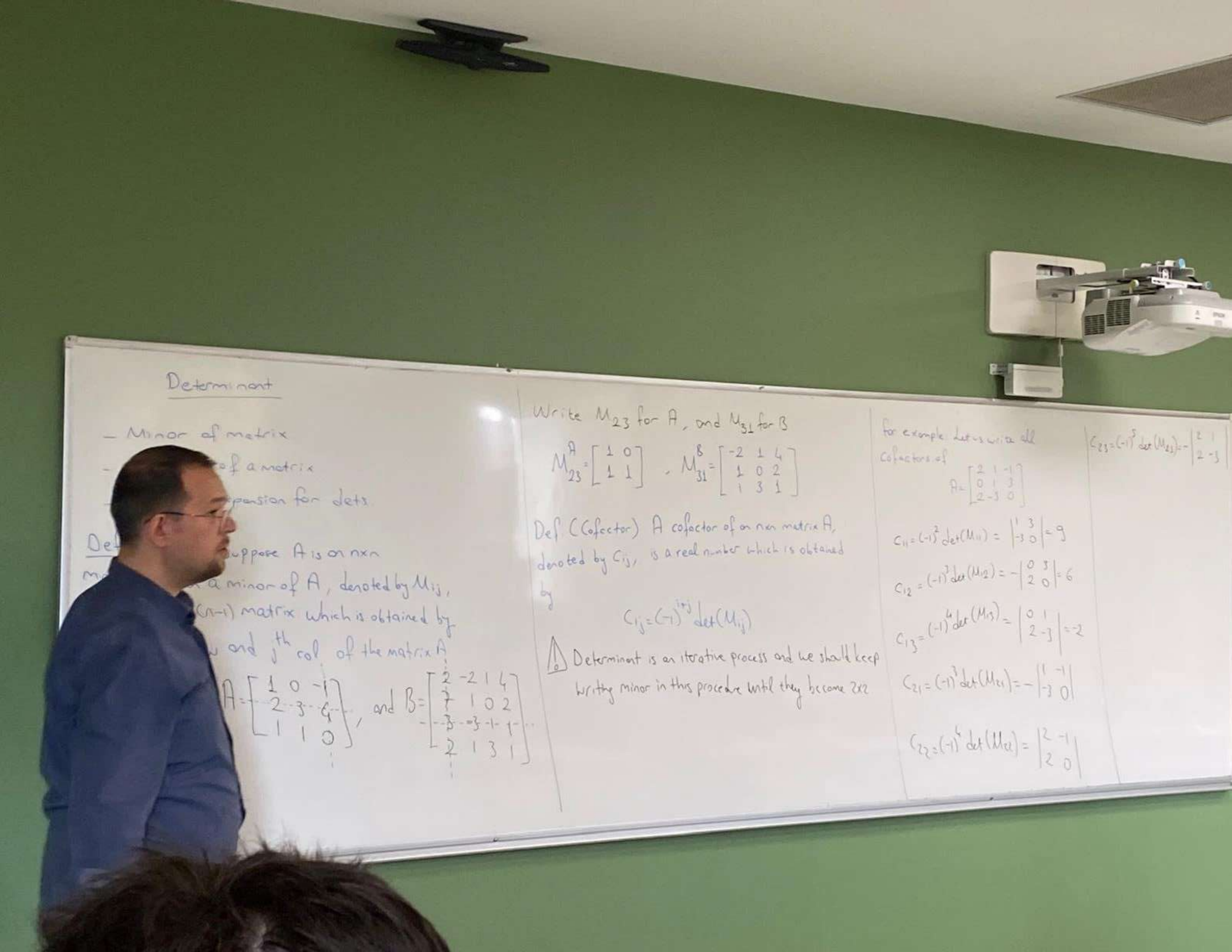
$Ax=0$ has non zero sol. (T is not 1-1)

Columns of A dont span $\mathbb{R}^3$ (T is not onto)

Let $L:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Is L onto and 1-1?


Determinant
– Minor of matrix
of a matrix
pansion for dets
Def
uppose A is an nxn
a minor of A, denoted by M_{ij},
(n-1) matrix which is obtained by
and j^{th} col of the matrix A
$A=\begin{bmatrix}1&0&-1\\2&3&4\\1&1&0\end{bmatrix}$, and $B=\begin{bmatrix}2&-2&1&4\\7&1&0&2\\3&3&1&1\\2&1&3&1\end{bmatrix}$
Write M_{23} for A, and M_{31} for B
$M^A_{23}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $M^B_{31}=\begin{bmatrix}-2&1&4\\1&0&2\\1&3&1\end{bmatrix}$
Def (Cofactor) A cofactor of an nxn matrix A, denoted by C_{ij}, is a real number which is obtained by
$C_{ij}=(-1)^{i+j}\det(M_{ij})$
⚠ Determinant is an iterative process and we should keep writing minor in this procedure until they become 2x2
for example Let's write all cofactors of
$A=\begin{bmatrix}2&1&-1\\0&1&3\\2&-3&0\end{bmatrix}$
$C_{11}=(-1)^2\det(M_{11})=\begin{vmatrix}1&3\\-3&0\end{vmatrix}=9$
$C_{12}=(-1)^3\det(M_{12})=-\begin{vmatrix}0&3\\2&0\end{vmatrix}=6$
$C_{13}=(-1)^4\det(M_{13})=\begin{vmatrix}0&1\\2&-3\end{vmatrix}=-2$
$C_{21}=(-1)^3\det(M_{21})=-\begin{vmatrix}1&-1\\-3&0\end{vmatrix}$
$C_{22}=(-1)^4\det(M_{22})=\begin{vmatrix}2&-1\\2&0\end{vmatrix}$
$C_{23}=(-1)^5\det(M_{23})=-\begin{vmatrix}2&1\\2&-3\end{vmatrix}$

Write M_{23} for A, and M_{31} for B.

$$M^{A}_{23} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M^{B}_{31} = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Def (Cofactor) A cofactor of an $n \times n$ matrix A, denoted by C_{ij}, is a real number which is obtained by

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

⚠ Determinant is an iterative process and we should keep writing minor in this procedure until they become 2x2

for example Let us write all cofactors of

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$C_{11} = (-1)^2 \det(M_{11}) = \begin{vmatrix} 1 & 3 \\ -3 & 0 \end{vmatrix} = 9$$

$$C_{12} = (-1)^3 \det(M_{12}) = -\begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$C_{13} = (-1)^4 \det(M_{13}) = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$C_{21} = (-1)^3 \det(M_{21}) = -\begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = -3$$

$$C_{22} = (-1)^4 \det(M_{22}) = \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{23} = (-1)^5 \det(M_{23}) = -\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = 8$$

$$C_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{32} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = -6$$

$$C_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

$$\left[\begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xleftrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{array}\right]$$

$(1-\gamma^2)$

No sol. case

$1-\gamma^2=0$ and $-1-\gamma \neq 0$

If $\gamma=1$, then no sol

Inf. many sol. case

$1-\gamma^2=0$ and $-1-\gamma=0$ / $\alpha \neq 0$

If $\gamma=-1$, then inf many sol's

Unique sol

$1-\gamma^2 \neq 0$, then $\gamma \neq \pm 1$ / $\alpha \neq 0$

Let S be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$S(x_1,x_2,x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - x_2 + 2x_3)$$

Write the kernel of S.

Singular

To answer this question

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2+R_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

If A is invertible, answer is Yes!

Otherwise, no!

$x_1 - x_2 + 2x_3 = 0$

$2x_1 + x_2 = 0$

$-x_1 - x_2 + 2x_3 = 0$

We need to solve $Ax=0$ if

$$A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{2}{3}R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4/3 & 0 \end{array}\right]$$

$\ker S = \{\vec{0}\}$

S is 1-1 and onto

Determinant

Def (Cofactor expansion) Let A be an $n \times n$ matrix

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} & & & & j^{th}\text{ col.} & & & \end{array} \\
 & \\
\begin{array}{c} \\ \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} &
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{ij} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\end{array}
$$

While evaluating the determinant, it is enough to pick one row or one column of a matrix

i^{th} row

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$= \sum_{k=1}^{n} a_{ik}C_{ik}$$

j^{th} col

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$= \sum_{k=1}^{n} a_{kj}C_{kj}$$

for example ...

Cofactors of

A ...

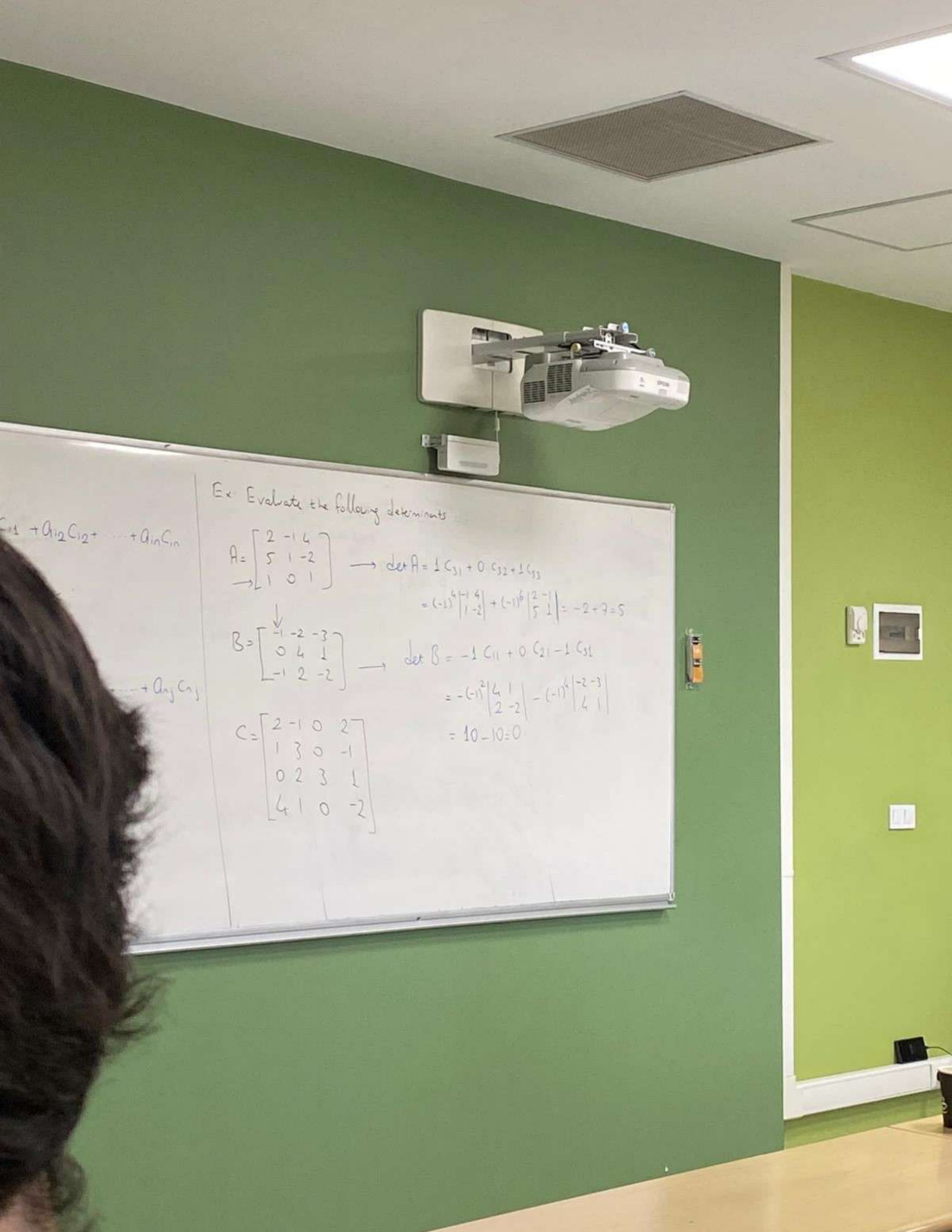
$C_{11} = (-1)^2 \det$ ...

$C_{12} = (-1)^3$ ...

$C_{13} = (-1)^4$ ...

$C_{21} = (-1$ ...

$C_{22} = (-$ ...



Ex: Evaluate the following determinants
+ a12 C12 + ... + a1n C1n
A = [2 -1 4; 5 1 -2; 1 0 1]
det A = 1 C31 + 0 C32 + 1 C33
= (-1)^4 |-1 4; 1 -2| + (-1)^6 |2 -1; 5 1| = -2 + 7 = 5
B = [-1 -2 -3; 0 4 1; -1 2 -2]
det B = -1 C11 + 0 C21 - 1 C31
= -(-1)^2 |4 1; 2 -2| - (-1)^4 |-2 -3; 4 1|
= 10 - 10 = 0
... + anj Cnj
C = [2 -1 0 2; 1 3 0 -1; 0 2 3 1; 4 1 0 -2]

$+ a_{in}C_{in}$

$\ldots + a_{nj}C_{nj}$

Ex Evaluate the following determinants

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \det A = 1\,C_{31} + 0\,C_{32} + 1\,C_{33}$$

$$= (-1)^4 \begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} + (-1)^6 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = -2 + 7 = 5$$

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 4 & 1 \\ -1 & 2 & -2 \end{bmatrix} \longrightarrow \det B = -1\,C_{11} + 0\,C_{21} - 1\,C_{31}$$

$$= -(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - (-1)^4 \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= 10 - 10 = 0$$

$$C = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 3 & 1 \\ 4 & 1 & 0 & -2 \end{bmatrix}$$

$$\det C = 3\,C_{33} = 3(-1)^6 \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= 3\left[2(-1)^2 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \right]$$

$$= 3\left[-10 - 2 - 22\right] = -102$$

Write M_{23} for A, and M_{31} for B

$$M_{23}^{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M_{31}^{B} = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Def (Cofactor) A cofactor of an $n \times n$ matrix A, denoted by C_{ij}, is a real number which is obtained by

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

⚠ Determinant is an iterative process and we should keep writing minor in this procedure until they become 2x2

for example Let us write all cofactors of

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$$C_{12} = (-1)^3 \det(M_{12}) = -\begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$C_{13} = (-1)^4 \det(M_{13}) = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$C_{21} = (-1)^3 \det(M_{21}) = -\begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = -3$$

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$$C_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$$

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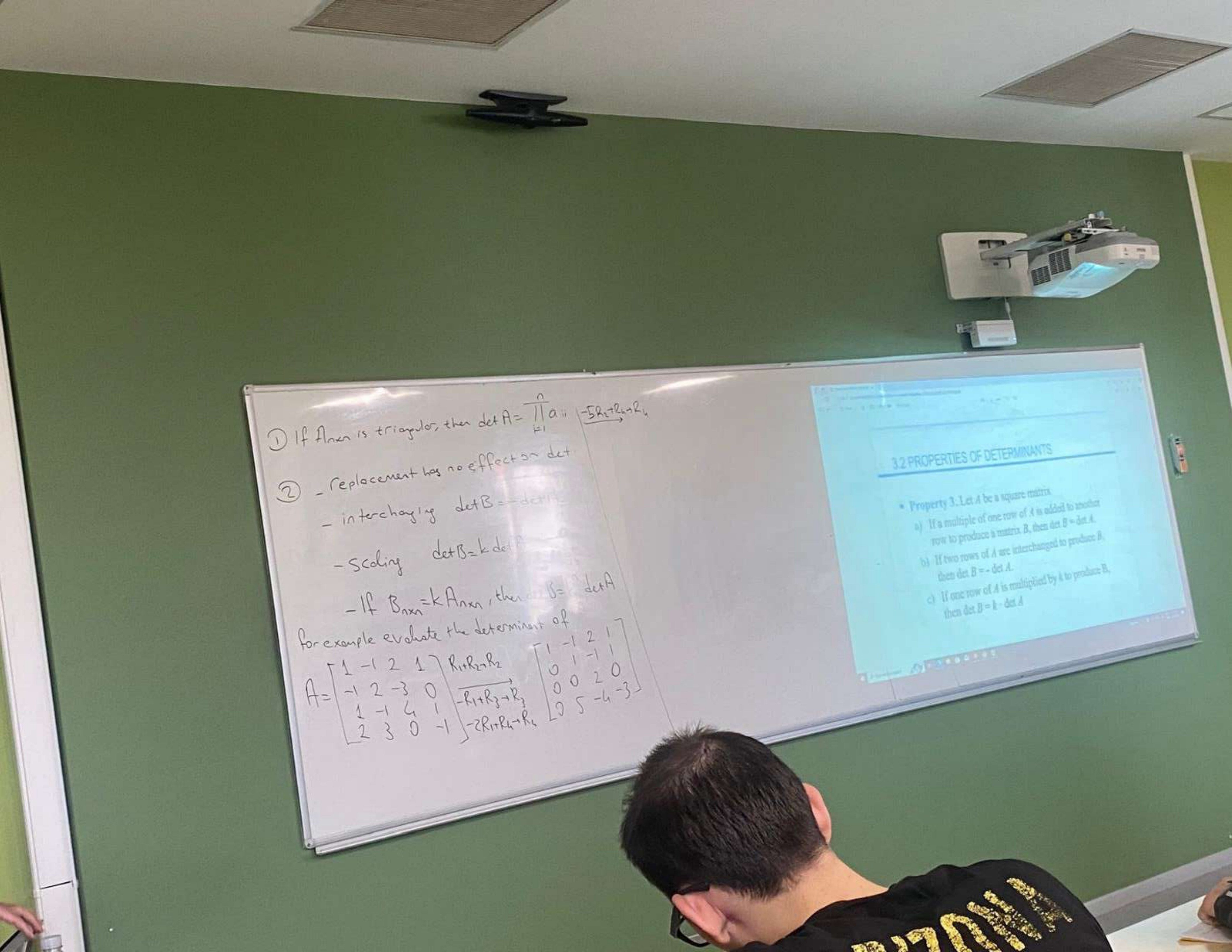
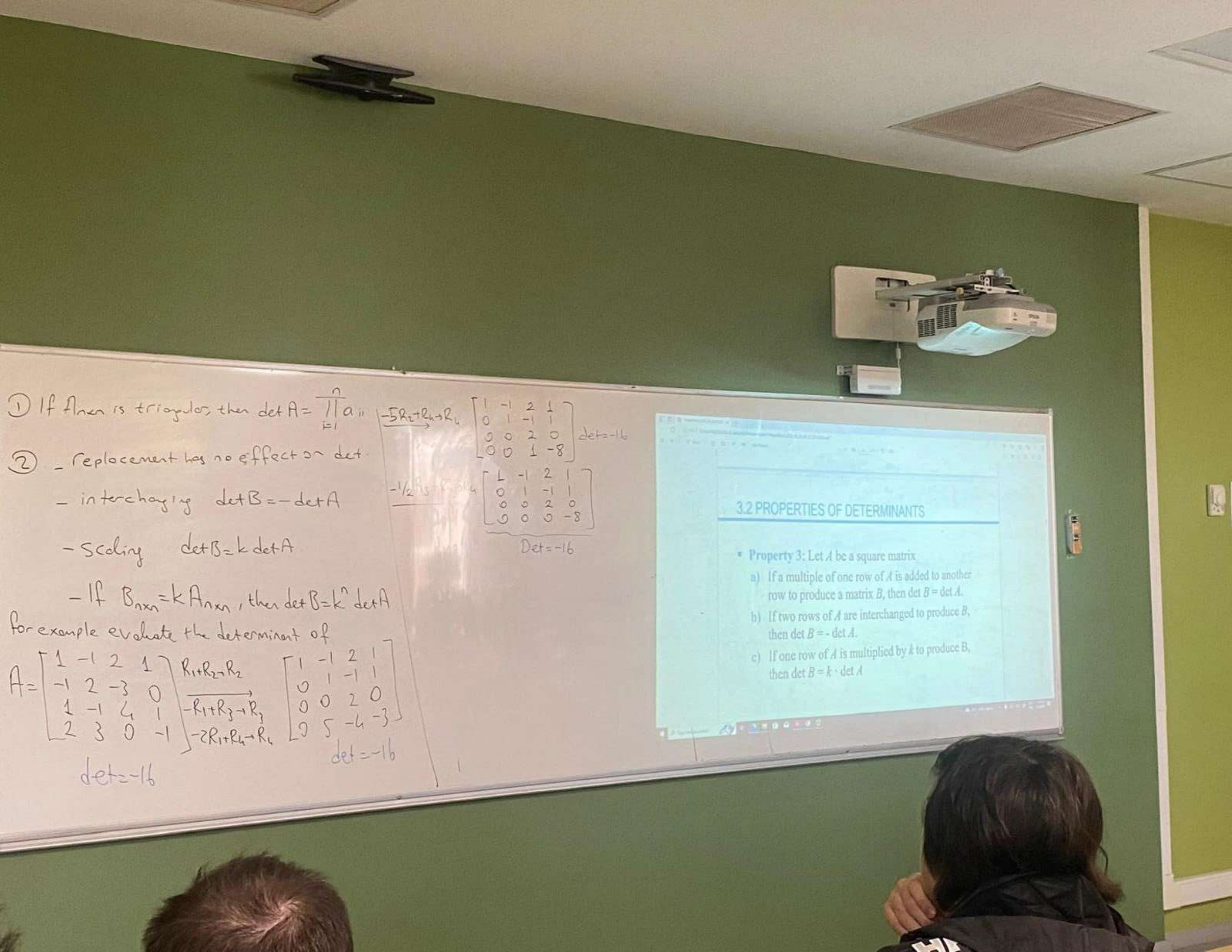
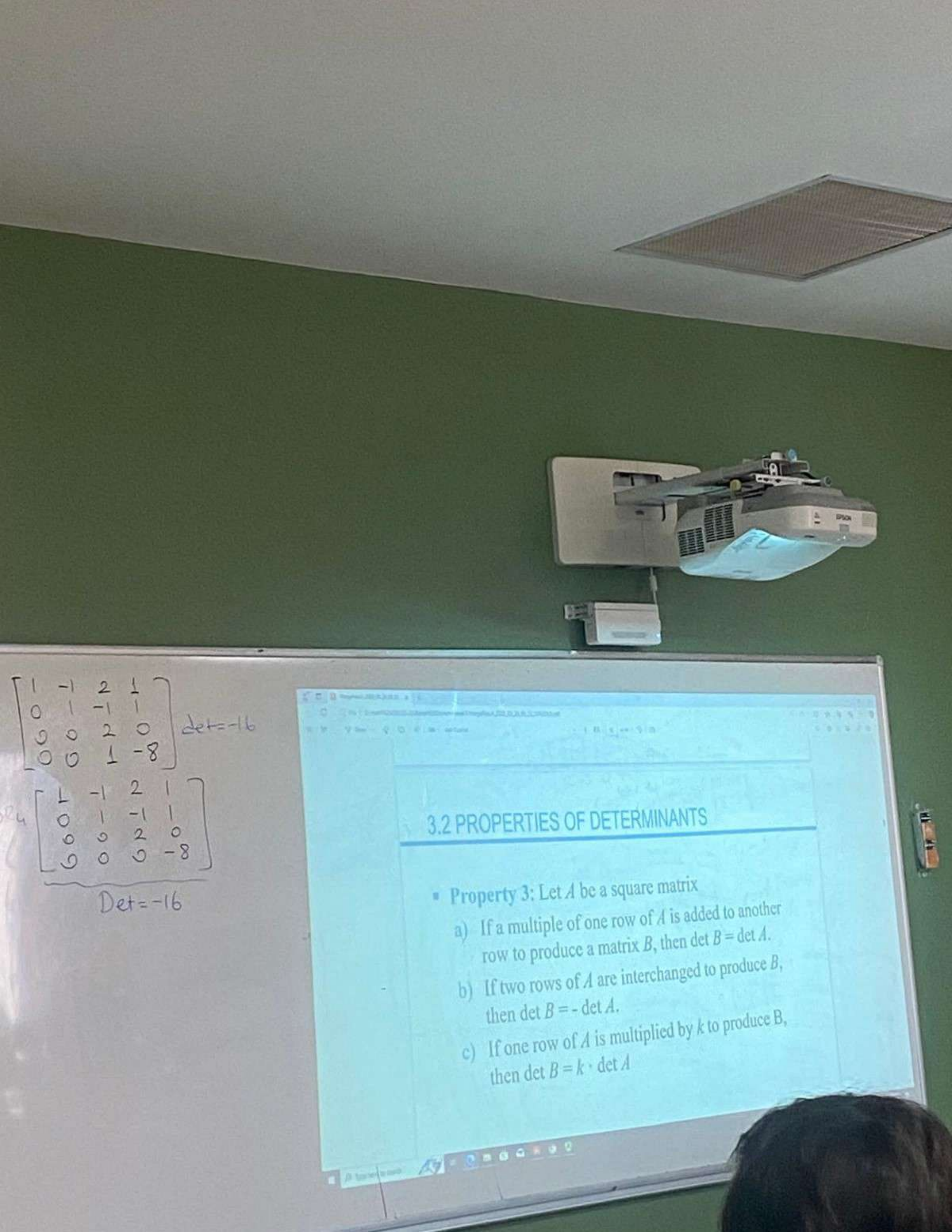

① If Anxn is triangular, then det A = ∏ aii
-5R2+R4→R4
② - replacement has no effect on det
- interchanging det B =
- scaling det B = k det
- If Bnxn = k Anxn, then
det A
for example evaluate the determinant of
A =
1 -1 2 1
-1 2 -3 0
1 -1 4 1
2 3 0 -1
R1+R2→R2
-R1+R3→R3
-2R1+R4→R4
1 -1 2 1
0 1 -1 1
0 0 2 0
0 5 -4 -3
3.2 PROPERTIES OF DETERMINANTS
Property 3. Let A be a square matrix
a) If a multiple of one row of A is added to another row to produce a matrix B, then det B = det A.
b) If two rows of A are interchanged to produce B, then det B = - det A.
c) If one row of A is multiplied by k to produce B, then det B = k · det A


① If $A_{n\times n}$ is triangular, then $\det A = \prod_{i=1}^{n} a_{ii}$
② - replacement has no effect on det
- interchanging $\det B = -\det A$
- scaling $\det B = k \det A$
- If $B_{n\times n} = k A_{n\times n}$, then $\det B = k^n \det A$
for example evaluate the determinant of
$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 0 \\ 1 & -1 & 4 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix}$
$R_1+R_2 \to R_2$
$-R_1+R_3 \to R_3$
$-2R_1+R_4 \to R_4$
$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 5 & -4 & -3 \end{bmatrix}$
det = -16
det = -16
$-5R_2+R_4 \to R_4$
$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix}$
det = -16
$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$
Det = -16
3.2 PROPERTIES OF DETERMINANTS
Property 3: Let A be a square matrix
a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.
c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$



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① If $A_{n\times n}$ is triangular, then $\det A = \prod_{i=1}^{n} a_{ii}$

② - replacement has no effect on det.

- interchanging $\det B = -\det A$

- scaling $\det B = k \det A$

- If $B_{n\times n} = k A_{n\times n}$, then $\det B = k^n \det A$

for example evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 0 \\ 1 & -1 & 4 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix} \xrightarrow[\substack{-R_1+R_3\to R_3 \\ -2R_1+R_4\to R_4}]{R_1+R_2\to R_2} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 5 & -4 & -3 \end{bmatrix}$$

det = -16   det = -16

$$\xrightarrow{-5R_2+R_4\to R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix} \quad \det = -16$$

$$\xrightarrow{-\frac{1}{2}R_3+R_4\to R_4} \underbrace{\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}}_{\text{Det} = -16}$$

$\ldots_2 C_{12} + \cdots + a_{in}C_{in}$

k

$+ a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

$\ldots_{ij} C_{kj}$

Ex. Evaluate the following determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \det A = 1\,C_{31} + 0\,C_{32} + 1\,C_{33}$$

$$= (-1)^4 \begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} + (-1)^6 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = -2 + 7 = 5$$

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 4 & 1 \\ -1 & 2 & -2 \end{bmatrix} \longrightarrow \det B = -1\,C_{11} + 0\,C_{21} - 1\,C_{31}$$

$$= -(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - (-1)^4 \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= 10 - 10 = 0$$

$$C = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ \cancel{0} & \cancel{2} & \cancel{3} & \cancel{1} \\ 4 & 1 & 0 & -2 \end{bmatrix}$$

$$\det C = 3\,C_{33} = 3(-1)^6 \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= 3\left[2(-1)^2 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}\right]$$

$$= 3[-10 - 2 - 22] = -102$$

Ex. Consider

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = W$$

Find

$$\begin{vmatrix} 2c & 2f & 2i \\ b+a & d+e & g+h \\ -a & -d & -g \end{vmatrix}$$

$$\underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{W} \xrightarrow{T} \underbrace{\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}}_{W} \xrightarrow{R_1+R_2\to R_2} \underbrace{\begin{vmatrix} a & d & g \\ b+a & e+d & g+h \\ c & f & i \end{vmatrix}}_{W} \xrightarrow{R_1\leftrightarrow R_3} \underbrace{\begin{vmatrix} c & f & i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix}}_{-W}$$

$$\xrightarrow{2R_1} \underbrace{\begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix}}_{-2W} \xrightarrow{-R_3} \underbrace{\begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix}}_{2W}$$

A + detB)

✓Cramer's rule

Adjoint of a matrix

Defn (Cramer's rule) This rule is used in the solutions of linear systems for which we are interested in only one particular unknown.

Suppose that A is an $n \times n$ matrix which is expressed interms of its columns so that

$$A = [C_1 \; C_2 \cdots C_i \cdots C_n]$$

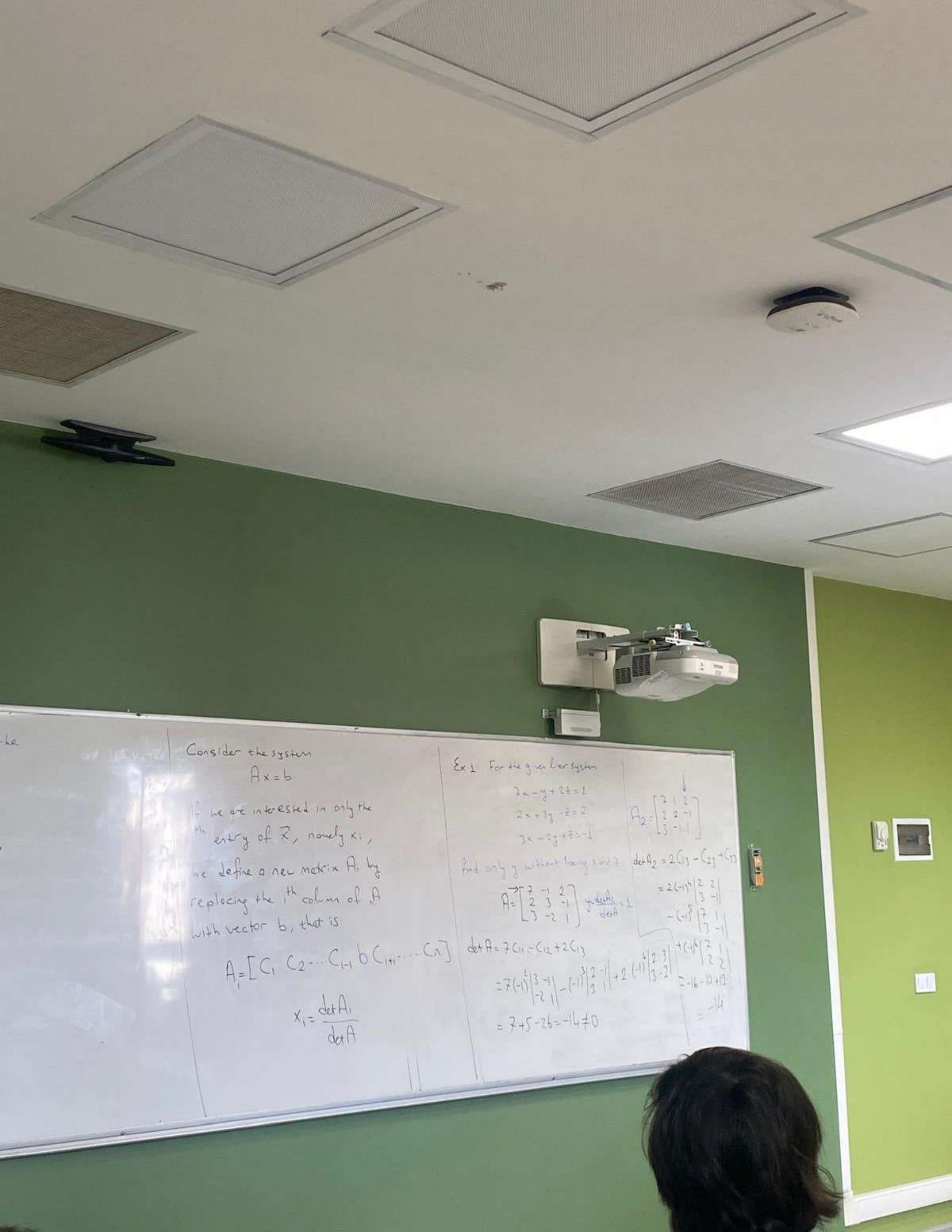
Consider the system

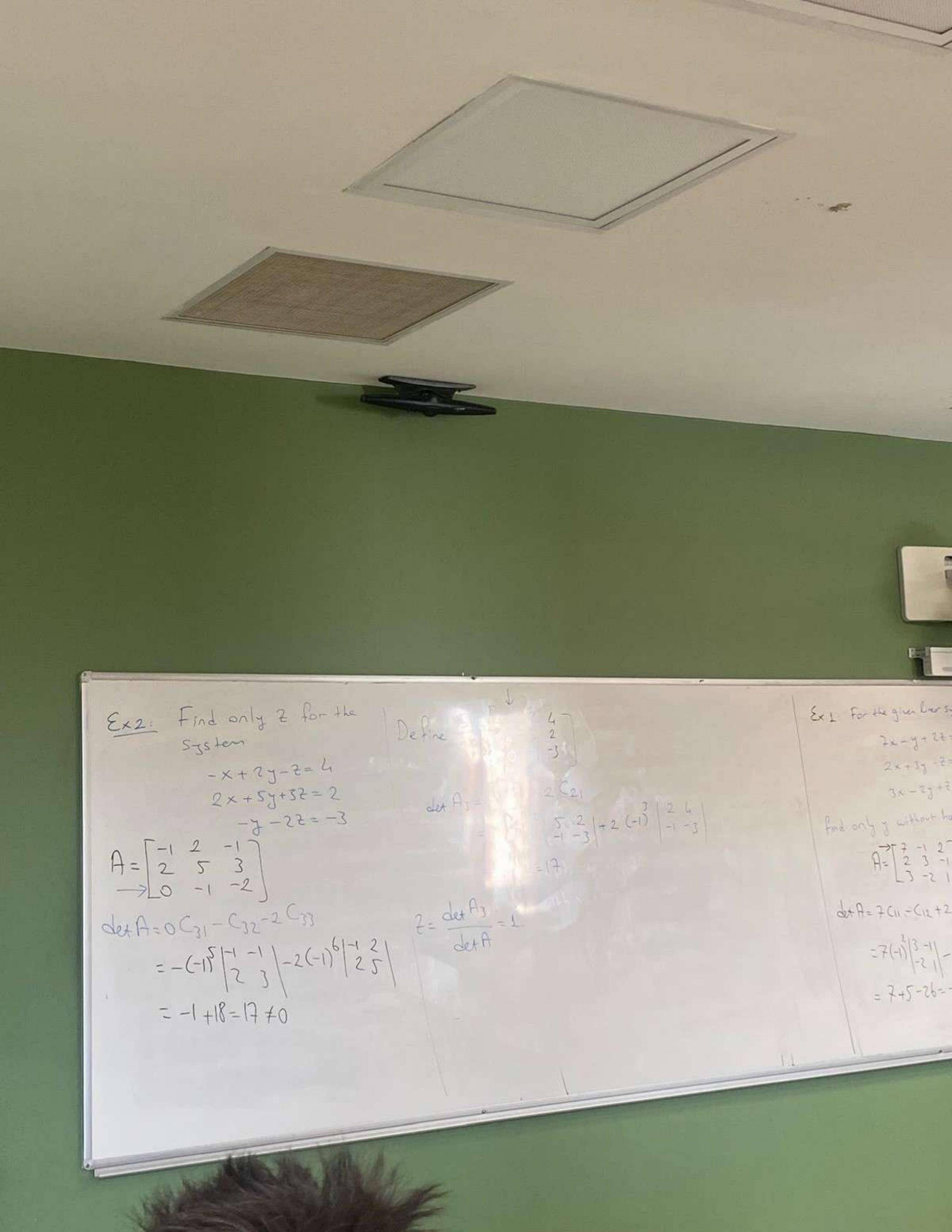
$$Ax = b$$

If we are interested in only the i^{th} entry of $\vec{x}$, namely x_i, we define a new matrix A_i by replacing the i^{th} column of A with vector b, that is

$$A_i = [C_1 \; C_2 \cdots C_{i-1} \; b \; C_{i+1} \cdots C_n]$$

$$x_i = \frac{\det A_i}{\det A}$$


Consider the system
Ax = b
If we are interested in only the ith entry of x, namely xi,
we define a new matrix Ai by
replacing the ith column of A
with vector b, that is
Ai = [C1 C2 ··· Ci-1 b Ci+1 ··· Cn]
xi = det Ai / det A
Ex 1 For the given linear system
7x - y + 2z = 1
2x + 3y - z = 2
3x - 2y + z = -1
Find only y without having x and z
y = det A2 / det A = 1
det A = 7C11 - C12 + 2C13
= 7 + 5 - 26 = -14 ≠ 0
det A2 = 2C13 - C23 + C33
= -16 - 10 + 12
= -14



Ex 2: Find only z for the system
$-x+2y-z=4$
$2x+5y+3z=2$
$-y-2z=-3$
$A=\begin{bmatrix}-1 & 2 & -1\\ 2 & 5 & 3\\ 0 & -1 & -2\end{bmatrix}$
$\det A = 0C_{31} - C_{32} - 2C_{33}$
$= -(-1)^5\begin{vmatrix}-1 & -1\\ 2 & 3\end{vmatrix} - 2(-1)^6\begin{vmatrix}-1 & 2\\ 2 & 5\end{vmatrix}$
$= -1 + 18 = 17 \neq 0$
Define
$\det A_3 =$
$= 17$
$z = \dfrac{\det A_3}{\det A} = 1$
Ex 1 For the given linear sy
find only y without
$\det A = 7C_{11} - C_{12} + 2$
$= 7 + 5 - 26 =$

Ex: Consider

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \omega$$

Find

$$\begin{vmatrix} 2c & 2f & 2i \\ b+a & d+e & g+h \\ -a & -d & -g \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{T} \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{vmatrix} a & d & g \\ b+a & e+d & g+h \\ c & f & i \end{vmatrix} \xrightarrow{R_1\leftrightarrow R_3} \begin{vmatrix} c & f & i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix}$$

$$\xrightarrow{2R_1} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix} \xrightarrow{-R_3} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix}$$

$tA + \det B)$

Define $A_3 = \begin{bmatrix} -1 & 2 & 4 \\ [illegible] & 5 & 2 \\ [illegible] \end{bmatrix}$

$\det A_3 = -$ [illegible]

$= -1\,(-1)$ [illegible] $\begin{vmatrix} \cdots & 2 \\ \cdots & 3 \end{vmatrix} + 2\,(-1)^3 \begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix}$

$= 13$

$z = \dfrac{\det A_3}{\det A} = 1.$

Ex Solve the

-he

$$A_1 = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 5 & 2 \\ 0 & -1 & -3 \end{bmatrix}$$

$$-1\,C_{11} + 2\,C_{21}$$

$$-1\cdot(-1)^2 \begin{vmatrix} 5 & 2 \\ -1 & -3 \end{vmatrix} + 2\,(-1)^3 \begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix}$$

$$= 13 + 4 = 17$$

3

$$t = \frac{[illegible]}{\det A} = 1$$

$$\begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix}$$

Linear Independence

Suppose that $\{v_1, v_2, \dots, v_n\}$ be a set of vectors. Then the given set is said to be a linearly independent set (the vectors are called linearly independent) if

$$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$$

holds for $c_1 = c_2 = \dots = c_n = 0$.

Remark 1. In $\mathbb{R}^n$, we can write at most n linearly independent vectors. Any $n+1$ vectors … linearly dependent.

Remark 2. Suppose that the set $\{v_1, \dots, v_n\}$ is linearly dependent. Then, there is at least one vector v_i in this set so that

$$v_i \in \operatorname{span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}.$$

Remark 3. From the perspective of linear systems, a set $\{v_1, v_2, \dots, v_n\}$ is called linearly independent set if and only if the system

$$A\vec{c} = 0$$

has trivial sol, where … $v_2 \dots v_n]$, and $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

For example

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$$

is linearly dependent since there are 4 vectors from $\mathbb{R}^3$.

For example:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Corresponding system is

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & -1 & -3 & 0 \end{array}\right] \xrightarrow[2R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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has trivial sol., where

$$A = [v_1 \, v_2 \cdots v_n], \text{ and } c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

$\{v_1, \dots, v_n\}$

here is at least

so that

$_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$.

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system

and $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

<u>for exemple</u> $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$

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$$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Corresponding system is

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & -1 & -3 & 0 \end{array}\right] \xrightarrow[2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$$

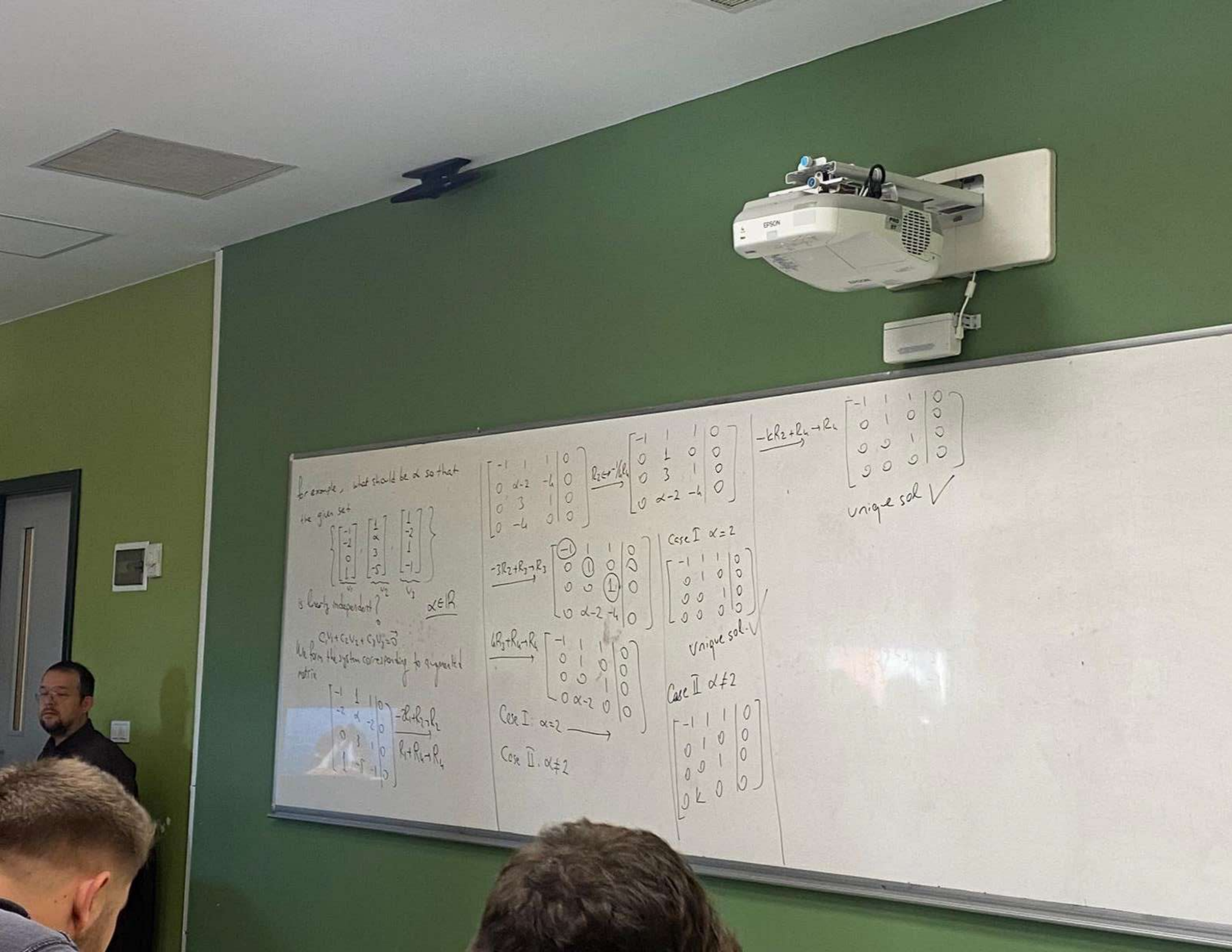
$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

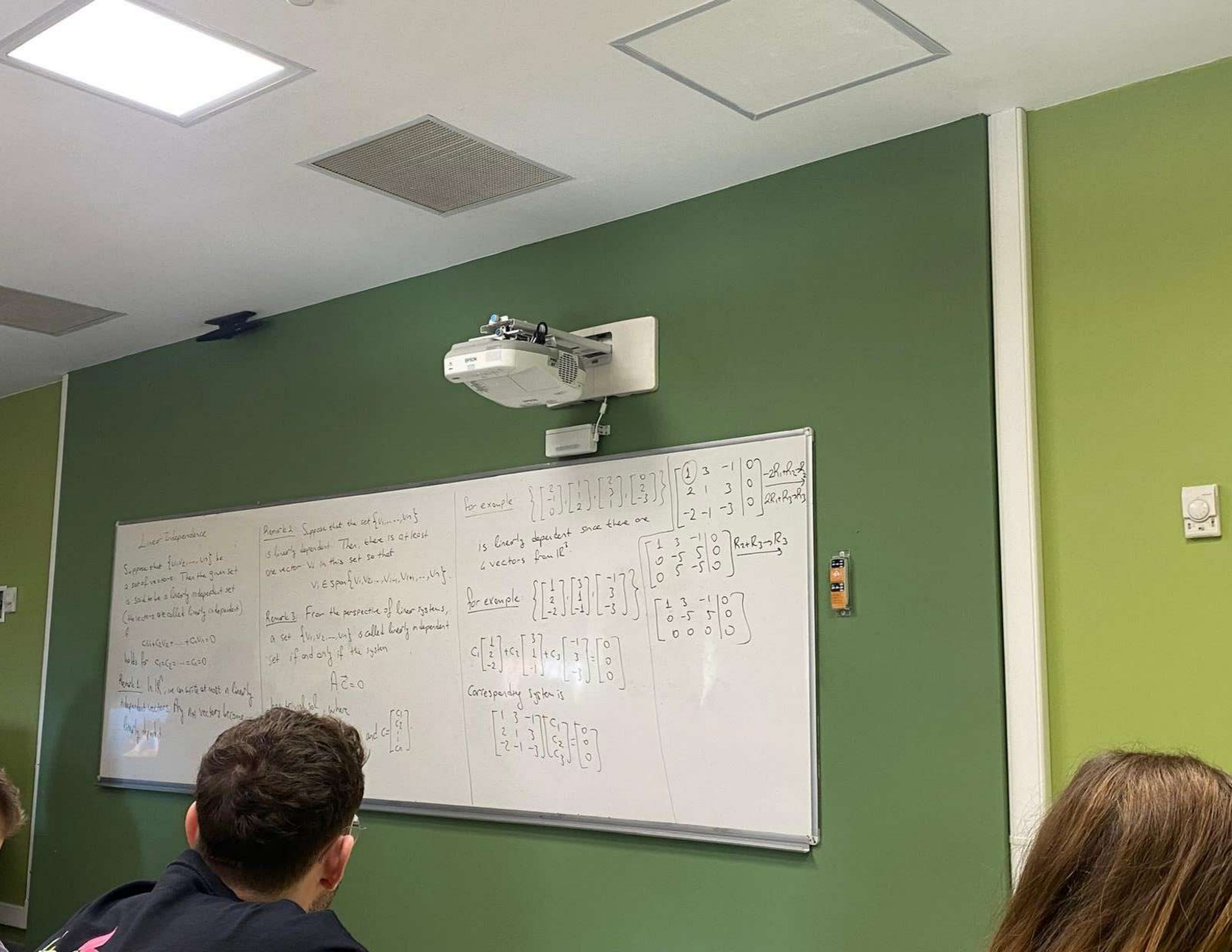
inf. many sol's

||

nonzero sol.

The set is linearly dependent.


for example, what should be α so that
the given set
is linearly independent?
α∈R
$c_1v_1+c_2v_2+c_3v_3=\vec{0}$
We form the system corresponding to augmented matrix
Case I α=2
unique sol.
Case II α≠2
Case I: α=2
Case II: α≠2
unique sol



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$v_i \in \text{span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$.
Remark 3 From the perspective of linear systems, a set $\{v_1, v_2, \dots, v_n\}$ is called linearly independent set if and only if the system
$A\vec{c} = 0$
where
and $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.
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$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Corresponding system is
$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & -1 & -3 & 0 \end{array}\right] \xrightarrow[2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2}$
$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3}$
$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

For example, what should be α so that the given set

$$\left\{ \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ \alpha \\ 3 \\ -5 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}}_{v_3} \right\}$$

is linearly independent? $\quad \underline{\alpha \in \mathbb{R}}$

$$c_1v_1 + c_2v_2 + c_3v_3 = \vec{0}$$

We form the system corresponding to augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -2 & \alpha & -2 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & -5 & -1 & 0 \end{array}\right] \xrightarrow[R_1+R_4 \to R_4]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow -\frac{1}{4}R_4} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array}\right]$$

$$\xrightarrow{4R_3+R_4 \to R_4} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \end{array}\right]$$

Case I: $\alpha = 2 \longrightarrow$

Case II: $\alpha \neq 2$

Case I $\alpha = 2$

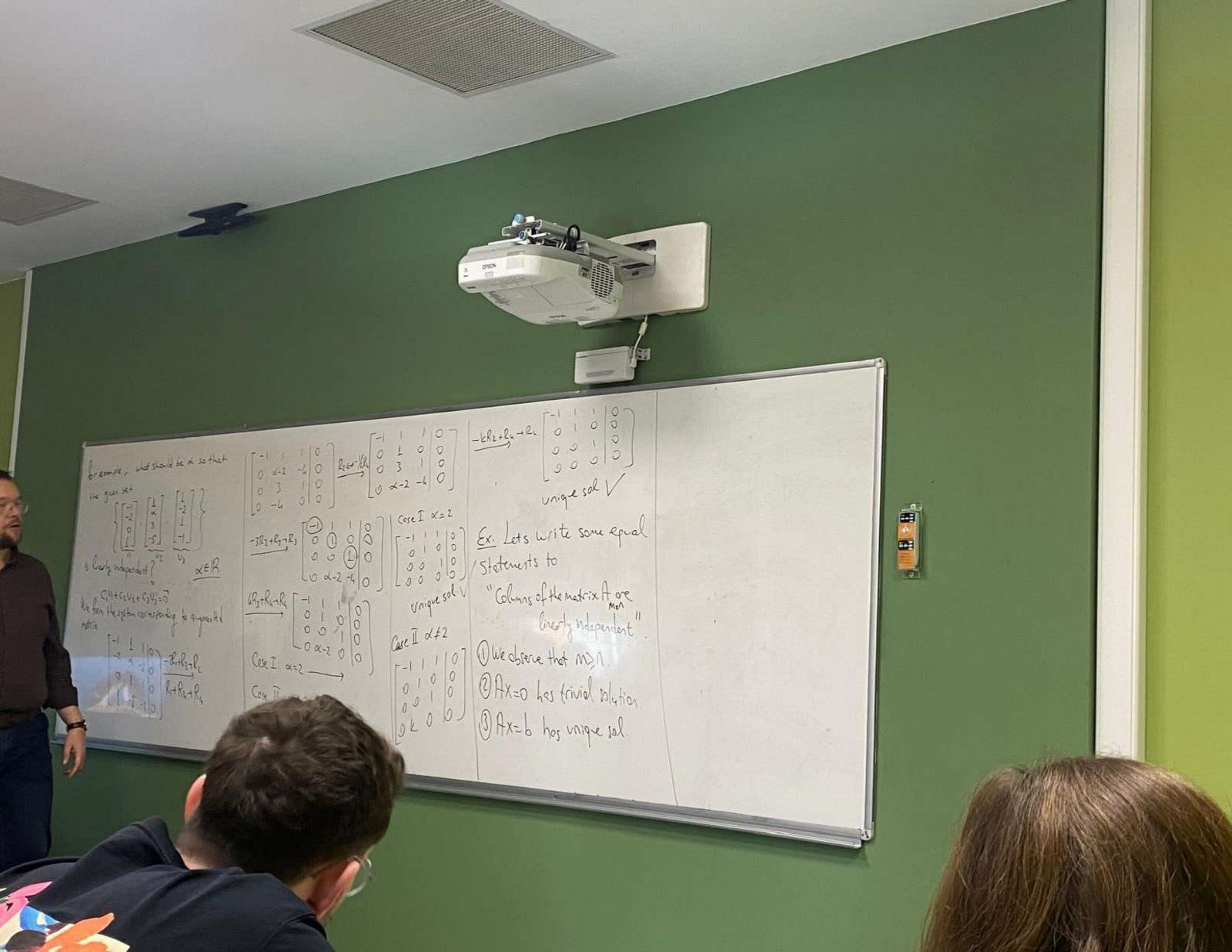
$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Unique sol. ✓

Case II $\alpha \neq 2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \end{array}\right] \xrightarrow{-kR_2+R_4 \to R_4} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

unique sol ✓


For example, what should be α so that
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α∈ℝ
We form the system corresponding to augmented matrix
Case I α=2
unique sol
Case II α≠2
unique sol
Ex. Lets write some equal
statements to
"Columns of the matrix A are
linearly independent"
① We observe that m≥n.
② Ax=0 has trivial solution
③ Ax=b has unique sol

For example, what should be α so that the given set

$$\left\{ \underbrace{\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1\\\alpha\\3\\-5\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}1\\-2\\1\\-1\end{bmatrix}}_{v_3} \right\}$$

is linearly independent? $\alpha \in \mathbb{R}$

$$c_1v_1 + c_2v_2 + c_3v_3 = \vec{0}$$

We form the system corresponding to augmented matrix

$$\left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ -2 & \alpha & -2 & 0\\ 0 & 3 & 1 & 0\\ 1 & -5 & -1 & 0\end{array}\right] \xrightarrow[R_1+R_4\to R_4]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & \alpha-2 & -4 & 0\\ 0 & 3 & 1 & 0\\ 0 & -4 & 0 & 0\end{array}\right] \xrightarrow{R_2\leftrightarrow -\frac{1}{4}R_4} \left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 3 & 1 & 0\\ 0 & \alpha-2 & -4 & 0\end{array}\right]$$

$$\xrightarrow{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \alpha-2 & -4 & 0\end{array}\right]$$

$$\xrightarrow{4R_3+R_4\to R_4} \left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \alpha-2 & 0 & 0\end{array}\right]$$

Case I: $\alpha = 2 \longrightarrow$

Case II: $\alpha \neq 2$

Case I $\alpha = 2$

$$\left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

unique sol. ✓

Case II $\alpha \neq 2$

$$\left[\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & k & 0 & 0\end{array}\right] \xrightarrow{-kR_2+R_4\to R_4}$$

Ex. Lets u

statements

"Columns

① We obse

② $Ax=0$

③ $Ax=$

$$\left[\begin{array}{c|c} \cdots & 0 \\ \cdots & 0 \\ \cdots & 0 \\ \cdots & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow -\frac{1}{4}R_4} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \end{array}\right] \longrightarrow$$

$\neq 2$

Case I $\alpha = 2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

unique sol. ✓

Case II $\alpha \neq 2$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \end{array}\right] \xrightarrow{-kR_2 + R_4 \to R_4} \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

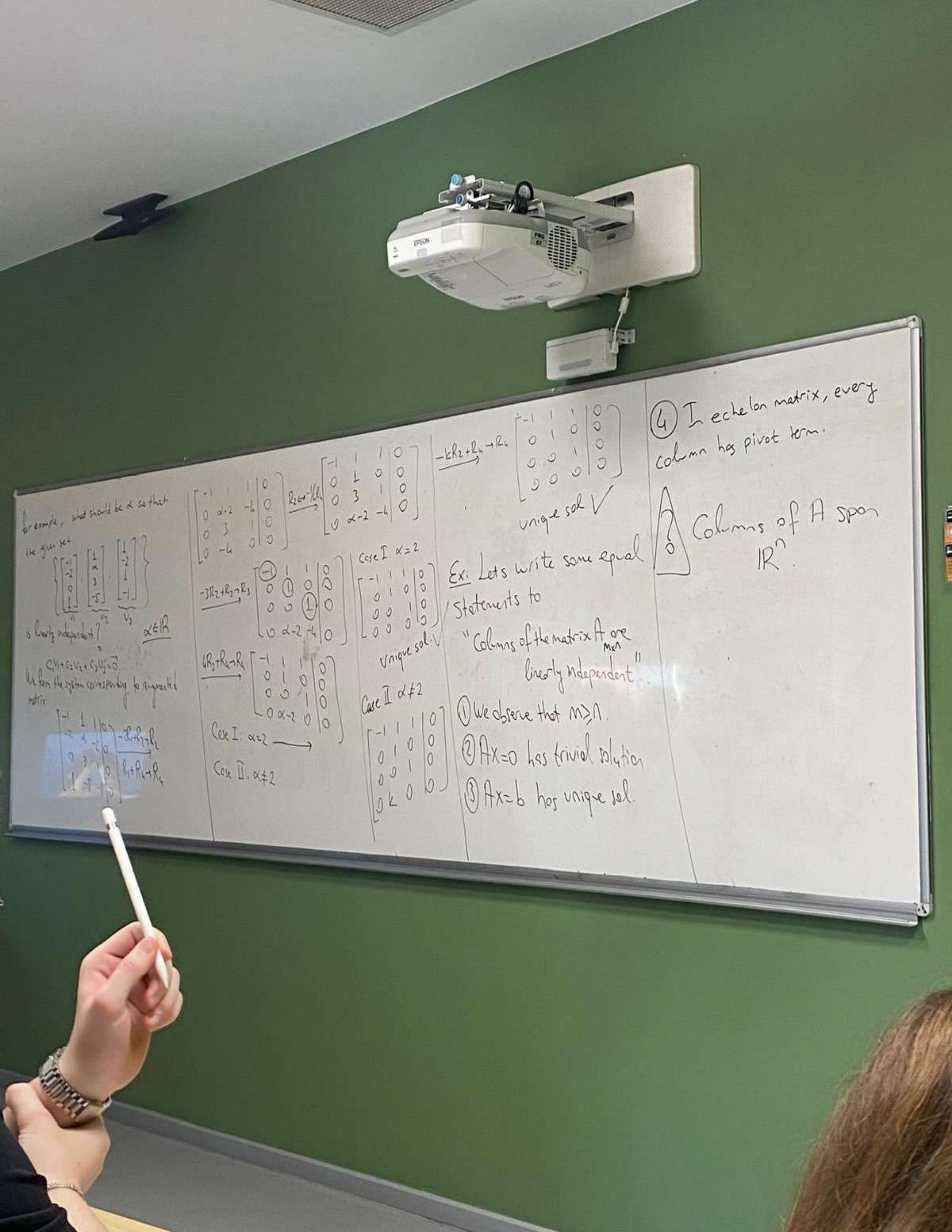
unique sol ✓

<u>Ex.</u> Lets write some equal statements to

"Columns of the matrix $A_{m\times n}$ are linearly independent"

1. We observe that $m \geq n$.
2. $Ax=0$ has trivial solution
3. $Ax=b$ has unique sol.
4. In echelon matrix, every column has pivot term.

⚠ Columns of A span $\mathbb{R}^n$.



for example, what should be α so that the given set
is linearly independent?
α∈IR
We form the system corresponding to augmented matrix
Case I. α=2
Case II. α≠2
Case I α=2
unique sol.
Case II α≠2
unique sol
Ex. Lets write some equal statements to
"Columns of the matrix A are linearly independent".
① We observe that m≥n.
② Ax=0 has trivial solution
③ Ax=b has unique sol.
④ In echelon matrix, every column has pivot term.
Columns of A span IR^n

Linear transformations

Matrix Calculus

Inverse of a matrix

<u>Def:</u> (1-1) Suppose that $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, T is said to be 1-1 if and only if $Tx = Ty$ when $x = y$

Alternatively, we can define being 1-1 in a better form. We need to define null space (kernel) of transformation.

<u>Def:</u> The null space of a transformation is a set given by

$$\text{Null } T \text{ or } \text{Ker } T = \{x \in \mathbb{R}^n : Tx = \vec{0}\}$$

⚠ T is 1-1 if and only if $\text{Ker}(T) = \{\vec{0}\}$.

<u>Ex 1</u> Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is T 1-1? $\boxed{\text{No}}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} 2x+y=0 \\ x-y=0 \end{matrix}\right\} \begin{matrix} x=0 \\ y=0 \end{matrix}$$

$$\text{Ker } T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span}\{ \ldots$$

<u>Ex 2</u> $T: \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

...1?

$\{\vec{0}\}$.

$)$

$=0$

$=0$

$\boxed{2?}$

pan $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \neq \{\vec{0}\}$

<u>Ex 2</u>: $T: \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation given by

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+2y\\2x-y\\x\\y\end{pmatrix}$$

Is T 1-1?

Lets write the matrix of transformation

$$A_{4\times 2} = \begin{bmatrix}1&2\\2&-1\\1&0\\0&1\end{bmatrix}$$

To have a 1-1 transformation, we need see $Ker(T) = \{\vec{0}\}$ Equivalently, we can check

does the system $Ax=0$ have trivial sol

We immediately form the augmented matrix

$$\left[\begin{array}{cc|c}1&2&0\\2&-1&0\\1&0&0\\0&1&0\end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cc|c}1&2&0\\0&-5&0\\0&-2&0\\0&1&0\end{array}\right]$$

⚠ T is 1-1 if and only if $\mathrm{Ker}(T) = \{\vec{0}\}$.

Ex 1: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is T 1-1? $\boxed{\text{No}}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2x+y&=0 \\ x-y&=0 \end{aligned}\right\} \begin{aligned} x&=0 \\ y&=0 \end{aligned} \qquad \boxed{z?}$$

$$\mathrm{Ker}\,T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{\vec{0}\}$$

Ex 2: $T: \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is T 1-1?

Lets write the matrix of transformation

$$A_{4\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transformation, we need see $\mathrm{Ker}(T) = \{\vec{0}\}$. Equivalently, we can check does the system $Ax=0$ have trivial sol

We immediately form the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$x_2 = 0$ and $x_1 = 0$

Trivial sol for $Ax=0$ ✓

$$\Downarrow$$

$$\mathrm{Ker}(T) = \{\vec{0}\}$$

T is 1-1

⚠ T is 1-1 if and only if $\mathrm{Ker}(T)=\{\vec{0}\}$.

Ex 1: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is T 1-1? $\boxed{\text{No}}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2x+y&=0 \\ x-y&=0 \end{aligned}\right\} \begin{aligned} x&=0 \\ y&=0 \end{aligned} \qquad \boxed{z?}$$

$$\mathrm{Ker}\,T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{\vec{0}\}$$

Ex 2: $T: \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is T 1-1?

Lets write the matrix of transformation

$$A_{4\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transformation, we need see $\mathrm{Ker}(T)=\{\vec{0}\}$. Equivalently, we can check does the system $Ax=0$ have trivial sol.

We immediately form the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ & & \\ & & \\ & & \end{array}\right]$$

mation

e write

$\xrightarrow{6/7 R_2 + R_3 \to R_3}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

T-F: If the columns of the standart matrix of linear transformation T are linearly independent, then T is 1-1 (T)

<u>Ex 1</u>: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Is T 1-1? $\boxed{\text{No}}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} 2x+y=0 \\ x-y=0 \end{array}\right\} \begin{array}{l} x=0 \\ y=0 \end{array} \quad \boxed{z?}$$

$$\text{Ker}\, T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{\vec{0}\}$$

<u>Ex 2</u>: $T: \mathbb{R}^2 \to \mathbb{R}^4$

transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} =$$

Is T 1-1?

Lets write the matrix

$$A_{4\times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transform

$\text{Ker}(T) = \{\vec{0}\}$ Equiv

Ex3: $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 3x - y + 2z \\ -x + 4y - 3z \end{pmatrix}.$$

Is T 1-1? $\boxed{\text{Yes}}$

The matrix of transformation

$$A_{3\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ -1 & 4 & -3 \end{bmatrix}$$

Then, we'll solve $Ax=0$, and therefor we write

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & -1 & 2 & 0 \\ -1 & 4 & -3 & 0 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 6 & 0 & 0 \end{array}\right] \xrightarrow{6/7 R_2 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

unique sol.
zero sol.

T-F: If the colu…
trasformation T are …

Ex1: Let $T: \mathbb{R}^3$…

$T($…

Is T 1-1?

$T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$…

Ker T = {…

Defn: A linear transformation T is said to be onto if codomain(T) = Range(T).

Equivalently, $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If the columns of the matrix A for T span $\mathbb{R}^m$, then T is onto!

for example: $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \\ -x-y-2z \end{pmatrix}$$

Is T onto?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

That is

$c_1\begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

That is

$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}-1\\1\\-1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\-2\end{bmatrix}=\begin{bmatrix}u\\v\\w\end{bmatrix}$$

We form the augmented matrix

$$\left[\begin{array}{ccc|c}1&-1&0&u\\0&1&1&v\\-1&-1&-2&w\end{array}\right]$$

We want that this system is consistent!

$$\xrightarrow{R_1+R_3\to R_3}\left[\begin{array}{ccc|c}1&-1&0&u\\0&1&1&v\\0&-2&-2&u+w\end{array}\right]\xrightarrow{2R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-1&0&u\\0&1&1&v\\0&0&0&u+2v+w\end{array}\right]$$

inconsistent

Since the columns could not span $\mathbb{R}^3$ and since we have 3 columns, this means columns are dependent So T is also not 1-1.

<u>for example</u> $T:\mathbb{R}^3\to\mathbb{R}^2$ given by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x-y+z\\-x+2z\end{pmatrix}$$

Is T 1-1 and onto?

<u>sol</u> The matrix of the transformation is

$$A=\begin{bmatrix}1&-1&1\\-1&0&2\end{bmatrix}$$

⚠ Can $Ax=0$ have unique sol?

Defn: A linear transformation T is said to be onto if $\text{codomain}(T) = \text{Range}(T)$.

$$\boxed{\text{Equivalently, } T: \mathbb{R}^n \to \mathbb{R}^m \text{ be a linear transformation. If the columns of the matrix } A \text{ for } T \text{ span } \mathbb{R}^m \text{, then } T \text{ is onto!}}$$

for example: $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \\ -x-y-2z \end{pmatrix}$$

Is T onto?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

That is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

<u>Defn</u>: A linear transformation T is said to be onto if $\text{codomain}(T) = \text{Range}(T)$.

Equivalently, $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If the columns of the matrix A for T span $\mathbb{R}^m$, then T is onto!

for example: $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ y + z \\ -x - y - 2z \end{pmatrix}$$

Is T onto? $\boxed{\text{No!}}$

That is

$$c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

We form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ -1 & -1 & -2 & w \end{array}\right]$$

We want that this system is consistent!

$$\xrightarrow{R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ 0 & -2 & -2 & u+w \end{array}\right] \xrightarrow{2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ 0 & 0 & 0 & u+2v+w \end{array}\right]$$

<u>inconsistent</u>

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<u>for example</u> $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y + z \\ -x + 2z \end{pmatrix}$$

Is T 1-1 and onto?

<u>sol</u>: The matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

⚠ Can $Ax = 0$ have unique sol? No!

Not 1-1!

To check onto property we pick an arbitrary vector $\begin{bmatrix} u \\ v \end{bmatrix}$ and write

$$c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$3\begin{bmatrix}0\\1\\-2\end{bmatrix} = \begin{bmatrix}u\\v\\w\end{bmatrix}$

gmented matrix

$\left.\begin{matrix} \\ \\ 2\end{matrix}\right| \begin{matrix}u\\v\\w\end{matrix}$ We want that this system is consistent!

$\begin{bmatrix} 0 & u \\ 1 & v \\ -2 & u+w \end{bmatrix} \xrightarrow{2R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ 0 & 0 & 0 & u+2v+w \end{array}\right]$

inconsistent

ns could not span $\mathbb{R}^3$ and since we have

means columns are dependent. So T is also not 1-1.

for example: $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x-y+z\\-x+2z\end{pmatrix}$$

Is T 1-1 and onto?

sol: The matrix of the transformation is

$$A = \begin{bmatrix}1 & -1 & 1\\-1 & 0 & 2\end{bmatrix}$$

? Can $Ax=0$ have unique sol?

T-F: If T is 1-1, then T is onto (F)

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \\ 2x \end{pmatrix}$$

Classwork: Is T 1-1 and onto?

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$$ not onto!

$\boxed{T \text{ is } 1\text{-}1}$

To check 1-1, $Ax=0$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \quad \begin{matrix} x_2=0 \\ x_1=0 \end{matrix}$$

Concluding remarks:

- If T is 1-1 and onto, then it is called bijection.

* If T is a bijection, then it has inverse

Matrix Calculus

We can express an $m \times n$ matrix A in the form $[a_{ij}]$, the element in i^{th} row and j^{th} col

$i=1,\ldots,m$, $j=1,\ldots,n$

Transpose: Let $A=[a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted by A^T, is an $n \times m$ matrix given by $A^T=[a_{ji}]$

⚠ Precisely, we make rows columns of the new matrix while we are taking transpose

Transpose
Scalar multip
Addition
Multiplication
Inverse

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \searrow A^T$$

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -4 & 7 \\ 2 & 0 \end{bmatrix}$$

Scalar multiplication: For any matrix $A_{m\times n} = [a_{ij}]$ we can define scalar multip.

$$cA_{m\times n} = [ca_{ij}] \text{ for } c \in \mathbb{R}.$$

(multiply all entries with c).

Like

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

then

$$4A = \begin{bmatrix} 8 & 4 & 0 \\ -4 & 12 & 8 \end{bmatrix}$$

⚠ $(cA)^T = cA^T$ for $c \in \mathbb{R}$

Matrix addition Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m\times n$ matrices.

Then

$$\underset{m\times n}{A} + \underset{m\times n}{B} = \underset{m\times n}{C} = [c_{ij}] = [a_{ij} + b_{ij}]$$

Addition is term by term.

⚠ Precisely, we make rows columns of the new matrix while we're taking transpose.

Transpose ✓
Scalar multip ✓
Addition
Multiplication
Inverse

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}_{2\times 3} \to A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -3 & 1 \end{bmatrix}_{3\times 2}$$

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}_{3\times 3} \to B^T = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$$C = \begin{bmatrix} 1 & 2 \\ -4 & 7 \\ 2 & 0 \end{bmatrix}_{3\times 2} \to C^T = \begin{bmatrix} 1 & -4 & 2 \\ 2 & 7 & 0 \end{bmatrix}_{2\times 3}$$

T-F: If T is 1-1, then T is onto (F)

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \\ 2x \end{pmatrix}$$

Classwork: Is T 1-1 and onto?

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$$ not onto!

$\boxed{T \text{ is } 1\text{-}1}$

To check 1-1, $Ax=0$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$x_2 = 0$
$x_1 = 0$

Concluding remarks:

- If T is 1-1 and onto, then it is called bijection.
- * If T is a bijection, then it has inverse

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We can express an $m \times n$ matrix A in the form $[a_{ij}]$, $i=1,\dots,m$, $j=1,\dots,n$

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Transpose: Let $A=[a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted by A^T, is an $n \times m$ matrix given by $A^T = [a_{ji}]$.

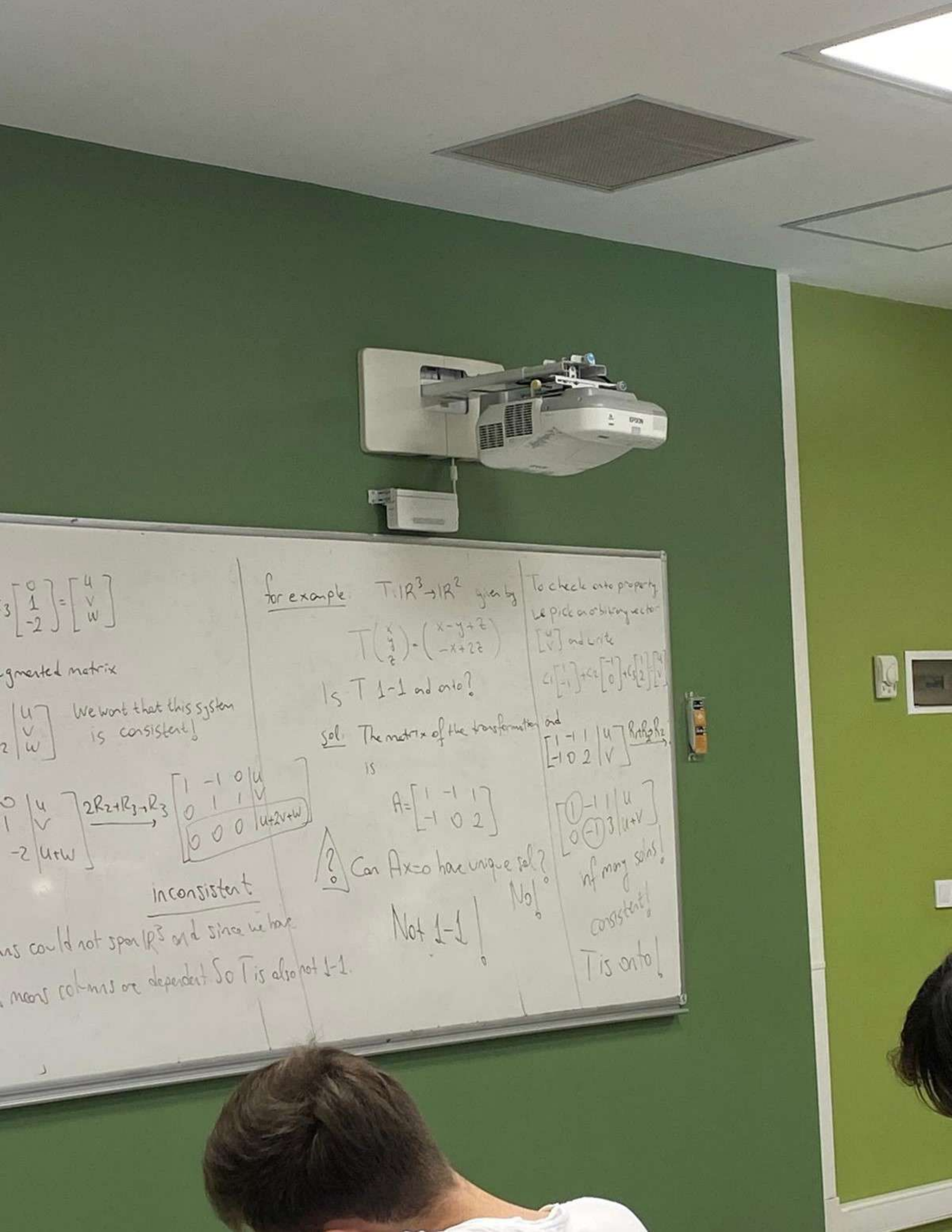
⚠ Precisely, we make rows columns of the new matrix while we are taking transpose.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$ $A^T =$

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -4 & 7 \\ 2 & 0 \end{bmatrix}$$

Transpose
Scalar multip
Addition
Multiplication
Inverse



gmented matrix
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Is T 1-1 and onto?
sol. The matrix of the transformation is
$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$
Can $Ax=0$ have unique sol?
No!
Not 1-1!
To check onto property we pick an arbitrary vector $\begin{bmatrix} u \\ v \end{bmatrix}$ and write
$\begin{bmatrix} 1 & -1 & 1 & u \\ -1 & 0 & 2 & v \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 1 & u \\ 0 & -1 & 3 & u+v \end{bmatrix}$
inf many solns!
consistent!
T is onto!

__Matrix addition__ Suppose that $A=[a_{ij}]$ and $B=[b_{ij}]$ be two $m\times n$ matrices.

Then

$$\underset{m\times n}{A}+\underset{m\times n}{B}=\underset{m\times n}{C}=[c_{ij}]=[a_{ij}+b_{ij}]$$

Addition is term by term.

$$\underbrace{\begin{bmatrix}2 & 3 & 1\\ -1 & 0 & 2\\ 4 & -1 & 2\end{bmatrix}_{3\times 3}}_{A}+\underbrace{\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -3\\ -3 & 2 & -1\end{bmatrix}_{3\times 3}}_{B}=\underbrace{\begin{bmatrix}1 & 1 & 4\\ 1 & 1 & -1\\ 1 & 1 & 1\end{bmatrix}_{3\times 3}}_{C}$$

$$(A+B)^T=A^T+B^T$$

Properties

$$c(A+B)=cA+cB$$

$$A+B=B+A$$

$$(A+B)+C=A+(B+C)$$

__Matrix Multiplication__

__Important:__ There is a very restrictive condition in matrix multip. according to sizes

$$A_{m\times \underline{n}}\,B_{\underline{n}\times p}=C_{m\times p}$$

__i.e__

$$A=\begin{bmatrix}2 & 1\\ 0 & 1\\ -1 & 2\end{bmatrix}$$

$$B=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 4\\ 2 & 1 & 7\end{bmatrix}$$

$$C=\begin{bmatrix}2 & -3 & -4\\ 0 & 1 & 1\end{bmatrix}$$

$$D=\begin{bmatrix}0 & 1 & 2 & -1\\ 2 & 4 & 2 & -1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$AB \neq BA$$

Transpose ✓
Scalar multip. ✓
Addition ✓
Multiplication
Inverse

$\underset{3\times 2}{A}\,\underset{3\times 3}{B}$ ✗

$\underset{3\times 3}{B}\,\underset{3\times 2}{A}$ ✓

$\underset{3\times 3}{B}\,\underset{2\times 3}{C}$ ✗

$\underset{2\times 3}{C}\,\underset{3\times 3}{B}$ ✓

$\underset{3\times 3}{B}\,\underset{3\times 4}{D}$ ✓

DB ✗

$\underset{3\times 2}{A}\,\underset{2\times 3}{C}$ ✓

$\underset{2\times 3}{C}\,\underset{3\times 2}{A}$ ✓

Scalar multiplication: For any matrix $A_{m\times n}=[a_{ij}]$ we can define scalar multip

$$cA_{m\times n}=[ca_{ij}] \text{ for } c\in\mathbb{R}.$$

(multiply all entries with c).

Like

$$A=\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

then

$$4A=\begin{bmatrix} 8 & 4 & 0 \\ -4 & 12 & 8 \end{bmatrix}$$

⚠ $(cA)^T=cA^T$ for $c\in\mathbb{R}$.

Matrix addition Suppose that $A=[a_{ij}]$ and $B=[b_{ij}]$ be two $m\times n$ matrices.

Then

$$\underset{m\times n}{A}+\underset{m\times n}{B}=\underset{m\times n}{C}=[c_{ij}]=[a_{ij}+b_{ij}]$$

Addition is term by term.

⚠ Precisely, we make rows columns of the new matrix while we are taking transpose.

Transpose ✓
Scalar multip ✓
Addition
Multiplication
Inverse

$$A=\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}_{2\times 3} \to A^T=\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -3 & 1 \end{bmatrix}_{3\times 2}$$

$$B=\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}_{3\times 3} \to B^T=\begin{bmatrix} 2 & 0 & 4 \\ -1 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$$C=\begin{bmatrix} 1 & 2 \\ -4 & 7 \\ 2 & 0 \end{bmatrix}_{3\times 2} \to C^T=\begin{bmatrix} 1 & -4 & 2 \\ 2 & 7 & 0 \end{bmatrix}_{2\times 3}$$

Computing AB

$$A_{m\times n} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \text{ where } R_i \text{ is a row of } A.$$

$$B_{n\times p} = [C_1 \ \ C_2 \ \cdots \ C_p], \text{ where } C_i \text{ is a column of } B.$$

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & \cdots & R_2C_p \\ \vdots & \vdots & & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & \cdots & R_mC_p \end{bmatrix}_{m\times p}$$

Ex:

$$BA = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{3\times 2}$$

Matrix addition

Suppose that $A=[a_{ij}]$ and $B=[b_{ij}]$ be two $m \times n$ matrices.

Then

$$\underset{m\times n}{A}+\underset{m\times n}{B}=\underset{m\times n}{C}=[c_{ij}]=[a_{ij}+b_{ij}]$$

Addition is term by term.

$$\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & -1 & 2 \end{bmatrix}_{3\times3}}_{A}+\underbrace{\begin{bmatrix} -1 & -2 & 3 \\ 2 & 1 & -3 \\ -3 & 2 & -1 \end{bmatrix}_{3\times3}}_{B}=\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3}}_{C}$$

⚠ $(A+B)^T=A^T+B^T$

Properties

$c(A+B)=cA+cB$

$A+B=B+A$

$(A+B)+C=A+(B+C)$

Matrix Multiplication

Important: There is a very restrictive condition in matrix multip. according to sizes.

$$\underset{m\times n}{A}\,\underset{n\times p}{B}=\underset{m\times p}{C}$$

i.e

$$A=\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B=\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}$$

$$C=\begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D=\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

⚠ $AB \neq BA$

Transpose ✓
Scalar multip. ✓
Addition ✓
Multiplication
Inverse

$\underset{3\times2}{A}\,\underset{3\times3}{B}$ ✗

$\underset{3\times3}{B}\,\underset{3\times2}{A}$ ✓

$\underset{3\times3}{B}\,\underset{2\times3}{C}$ ✗

$\underset{2\times3}{C}\,\underset{3\times3}{B}$ ✓

$\underset{3\times3}{B}\,\underset{3\times4}{D}$ ✓

DB ✗

$\underset{3\times2}{A}\,\underset{2\times3}{C}$ ✓

$\underset{2\times3}{C}\,\underset{3\times2}{A}$ ✓

Computing AB

$A_{m\times n} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$, where R_i is a row of A.

$B_{n\times p} = [C_1 \;\; C_2 \cdots C_p]$, where C_i is a column of B.

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & \cdots & R_2C_p \\ \vdots & \vdots & & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & \cdots & R_mC_p \end{bmatrix}_{m\times p}$$

Ex:

$$BA = \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 3 & 1 \\ -4 & 9 \\ -3 & 17 \end{bmatrix}_{3\times 2}$$

$$CA = \underbrace{\begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 8 & -9 \\ -1 & 3 \end{bmatrix}_{2\times 2}$$

$$BD = \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{3\times 4} = \begin{bmatrix} 4 & 9 & 5 & -5 \\ 2 & 4 & 6 & 7 \\ 2 & 6 & 13 & 11 \end{bmatrix}_{3\times 4}$$

Properties

$c(A+B) = cA + cB$

$A+B = B+A$

$(A+B)+C = A+(B+C)$

Matrix Multiplication

Important: There is a very restrictive condition in matrix multip. according to sizes.

$$A_{m\times \underline{n}} \, B_{\underline{n}\times p} = C_{m\times p}$$

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$D = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\triangle$ $AB \neq BA$

Transpose ✓

Scalar multip. ✓

Addition ✓

Multiplication

Inverse

$\underset{3\times 2}{A}\,\underset{3\times 3}{B}$ ✗

$\underset{3\times 3}{B}\,\underset{3\times 2}{A}$ ✓

$\underset{3\times 3}{B}\,\underset{2\times 3}{C}$ ✗

$\underset{2\times 3}{C}\,\underset{3\times 3}{B}$ ✓

$\underset{3\times 3}{B}\,\underset{3\times 4}{D}$ ✓

DB ✗

$\underset{3\times 2}{A}\,\underset{2\times 3}{C}$ ✓

$\underset{2\times 3}{C}\,\underset{3\times 2}{A}$ ✓

- Inverse of a matrix
- LU - factorization
- Determinants

} Slides

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Identity matrix $= I_{n\times n} = \begin{cases} a_{ij} = 1 & \text{if } i = j \\ a_{ij} = 0 & \text{if } i \neq j \end{cases}$

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

A^{-1} = inverse of A

invertible $\equiv$ nonsingular

noninvertible $\equiv$ singular

If $B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, then

$\det B = 2\cdot 1 - (-1)3 = 5$

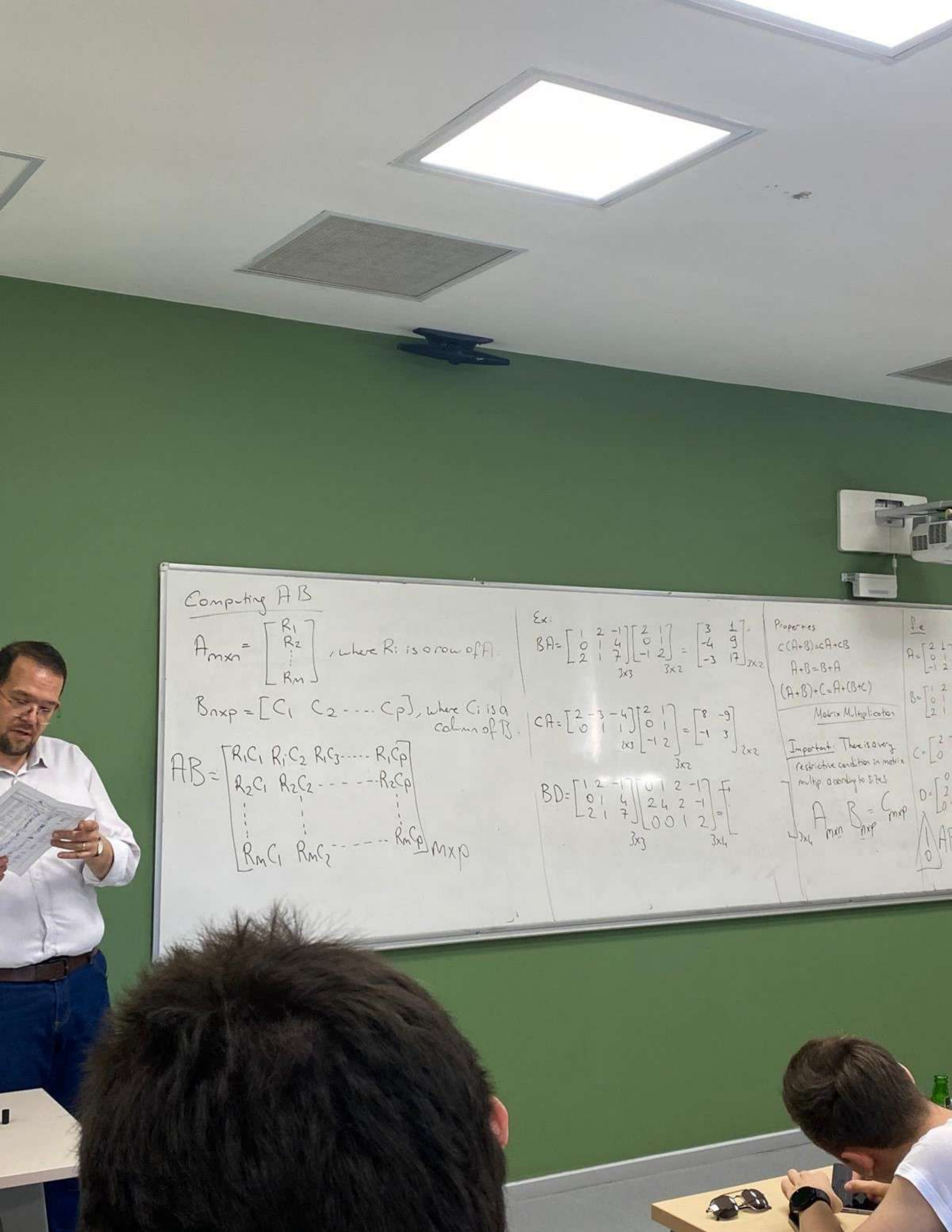
$$B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix}$$

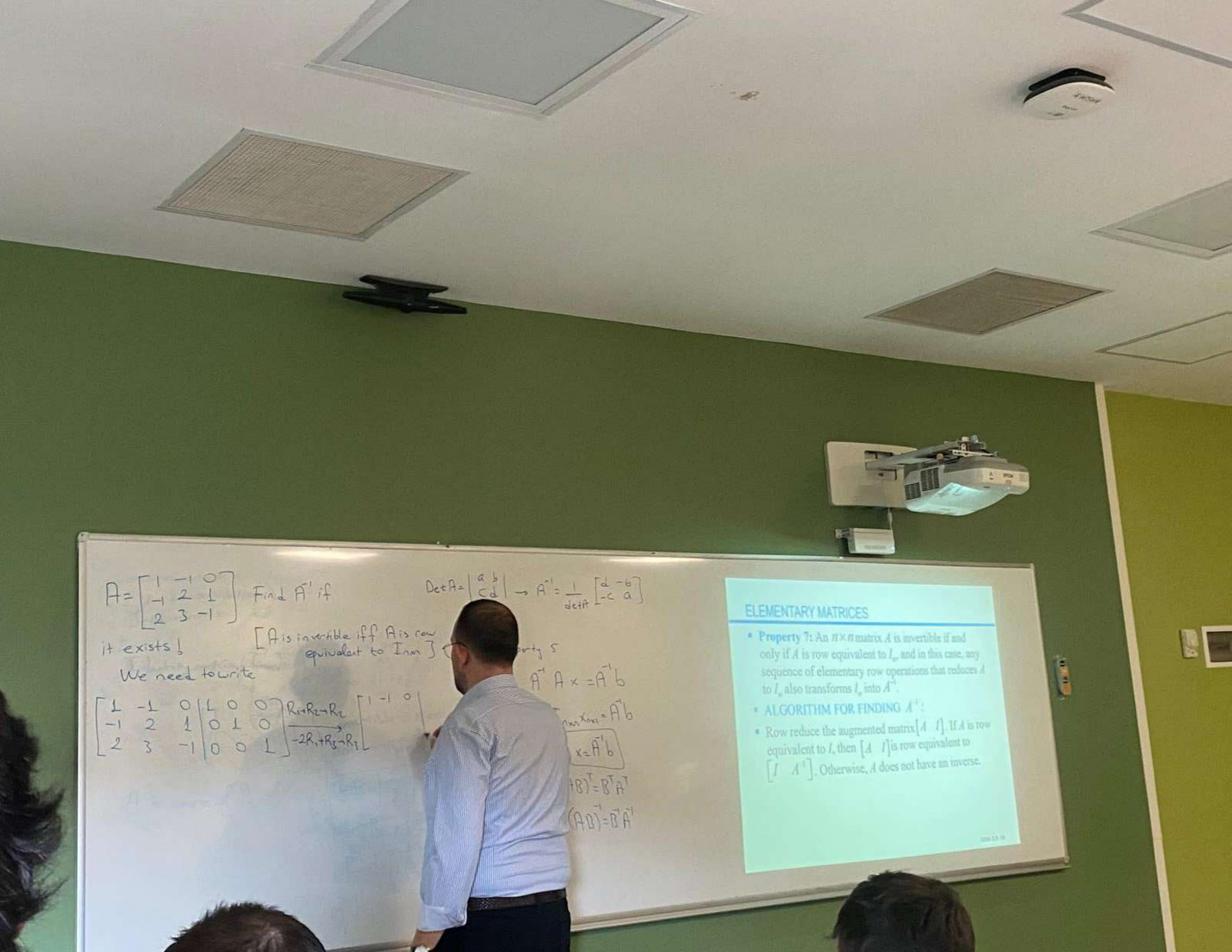
Property 5

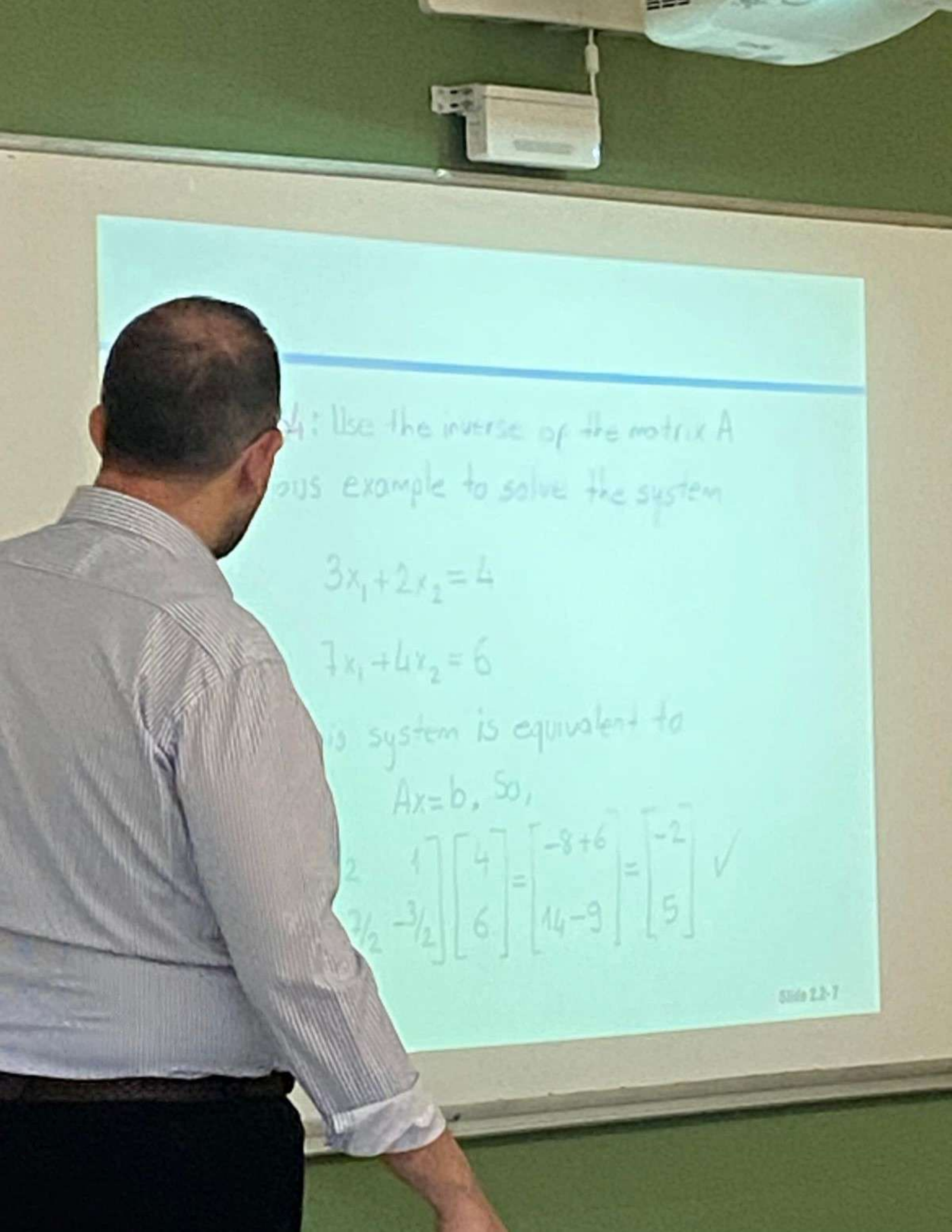
$$A^{-1}Ax = A^{-1}b$$

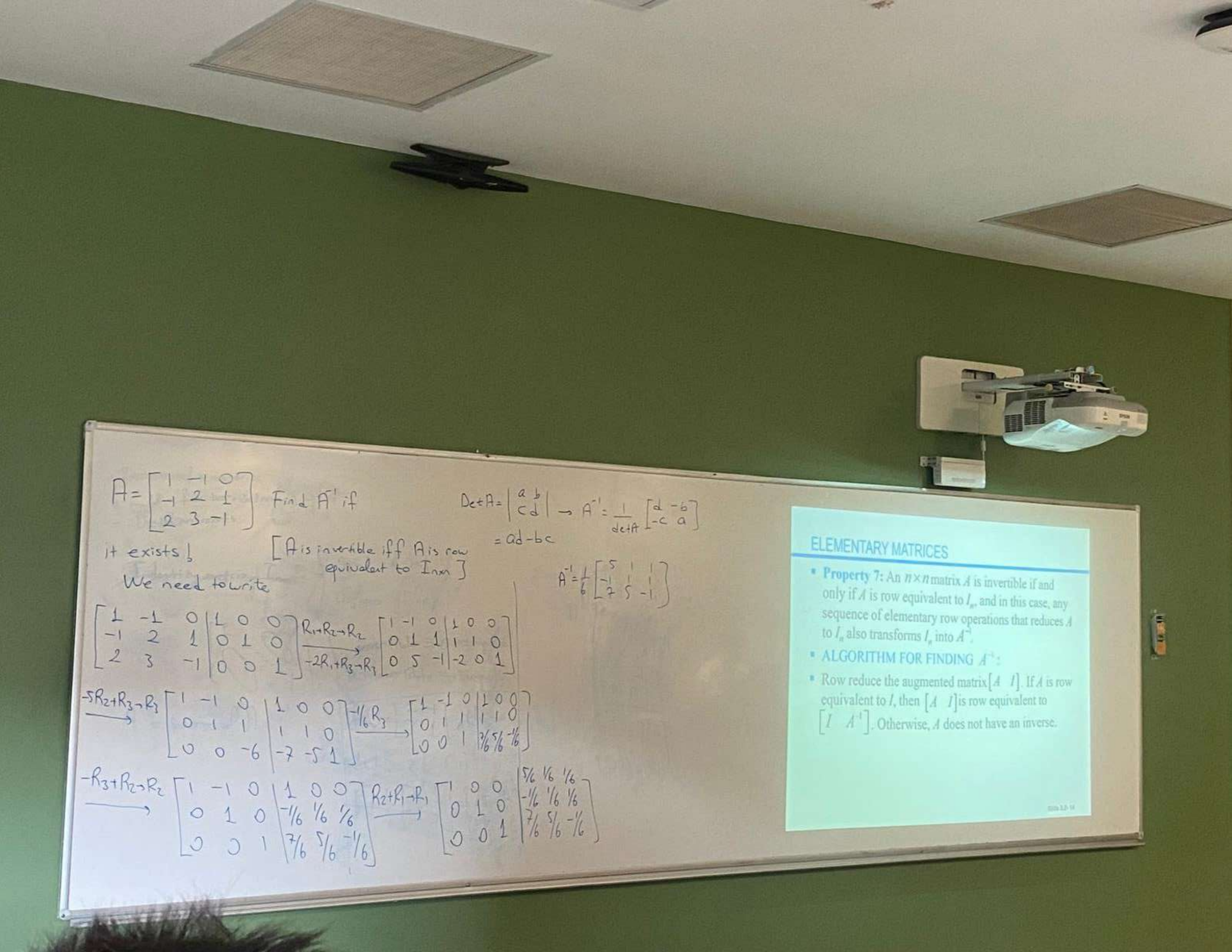
$$I_{n\times n}x_{n\times 1} = A^{-1}b$$

$$\boxed{x = A^{-1}b}$$


Computing AB
where Ri is a row of A
where Ci is a column of B.
Ex:
Properties
c(A+B)=cA+cB
A+B=B+A
(A+B)+C=A+(B+C)
Matrix Multiplication
Important: There is a very restrictive condition in matrix multip. according to sizes


ELEMENTARY MATRICES
Property 7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.
ALGORITHM FOR FINDING A^{-1}:
Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.
$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ Find A^{-1} if it exists!
$\text{Det} A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \to A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
[A is invertible iff A is row equivalent to $I_{n \times n}$]
We need to write
$A^{-1} A x = A^{-1} b$
$x = A^{-1} b$
$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2}$


4: Use the inverse of the matrix A
ous example to solve the system
$3x_1+2x_2=4$
$7x_1+4x_2=6$
is system is equivalent to
$Ax=b$, So,
$\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}-8+6\\14-9\end{bmatrix}=\begin{bmatrix}-2\\5\end{bmatrix}$ ✓
Slide 2.2-7



$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ Find A^{-1} if it exists!
$\text{Det}A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
[A is invertible iff A is row equivalent to $I_{n\times n}$]
We need to write
$A^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 7 & 5 & -1 \end{bmatrix}$
$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & -2 & 0 & 1 \end{array}\right]$
$\xrightarrow{-5R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & -7 & -5 & 1 \end{array}\right] \xrightarrow{-1/6 R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right]$
$\xrightarrow{-R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right]$
ELEMENTARY MATRICES
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$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ Find A^{-1} if it exists!

$\operatorname{Det} A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \to A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

[A is invertible iff A is row equivalent to $I_{n\times n}$]

We need to write

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-5R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & -7 & -5 & 1 \end{array}\right] \xrightarrow{-\frac{1}{6}R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right]$$

$$\xrightarrow{-R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array}\right]$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 7 & 5 & -1 \end{bmatrix}$$

Suppose that $A_{n\times n}$ is invertible. Then

i) A is row equivalent to $I_{n\times n}$

ii) $Ax=b$ has unique sol

iii) $Ax=0$ has trivial sol.

iv) A has n pivot terms

v) Columns of A span $\mathbb{R}^n$

vi) Columns of A are linearly independent

vii) The linear transformation $Tx = Ax$ is 1-1 and onto (by (iii) and (v))

viii) The linear transformation $Tx = Ax$ has inverse, where $T^{-1}x = A^{-1}x$

$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ Find A^{-1} if it exists!

[A is invertible iff A is row equivalent to $I_{n \times n}$]

We need to write

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-5R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & -7 & -5 & 1 \end{array}\right]$$

$\operatorname{Det} A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \to A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Property 5

$A^{-1} A x = A^{-1} b$

$I_{n \times n} x_{n \times 1} = A^{-1} b$

$\boxed{x = A^{-1} b}$

Remember: $(AB)^T = B^T A^T$

Similarly $(AB)^{-1} = B^{-1} A^{-1}$

ELEMENTARY MATRICES

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$^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 7 & 5 & -1 \end{bmatrix}$

Suppose that $A_{n\times n}$ is invertible.
Then

i) A is row equivalent to $I_{n\times n}$

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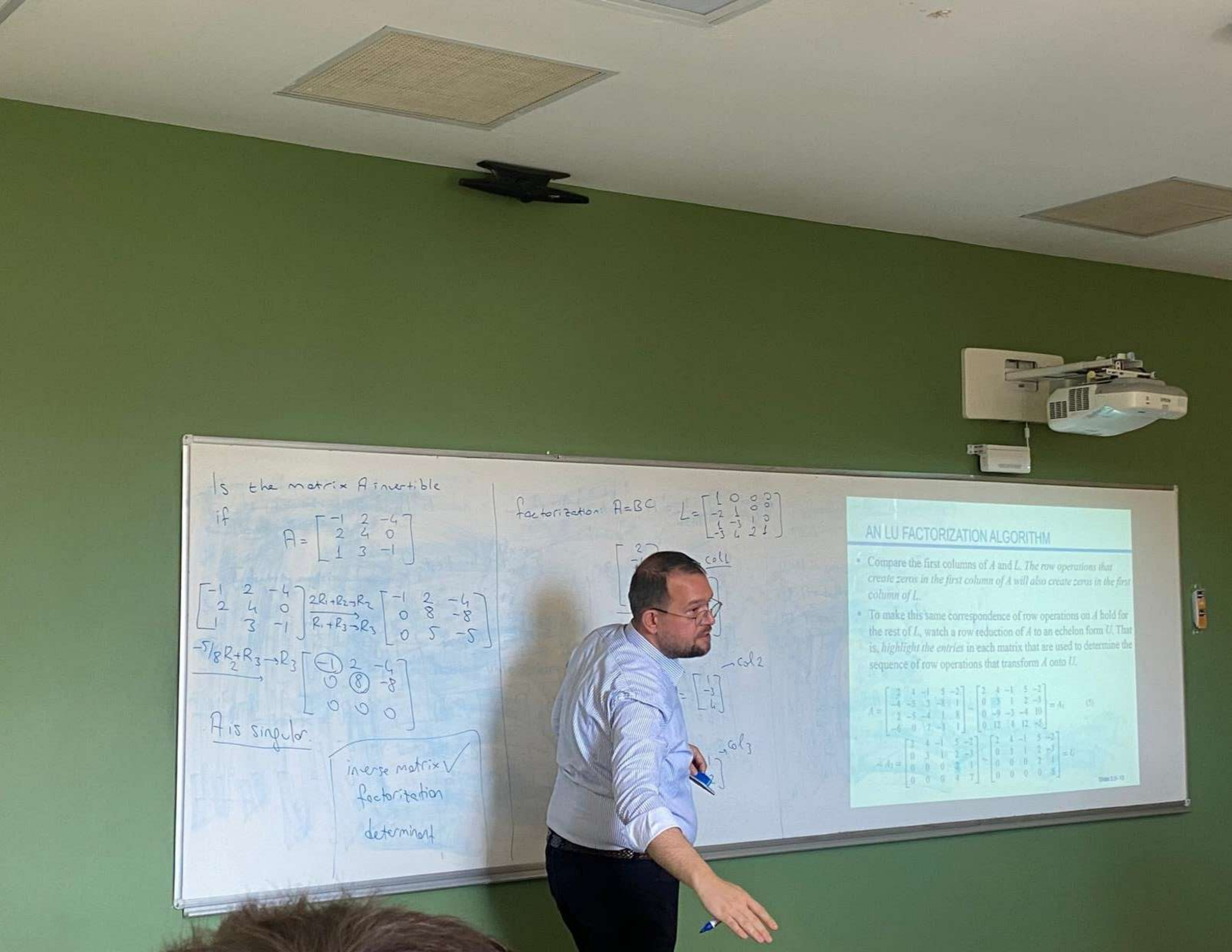
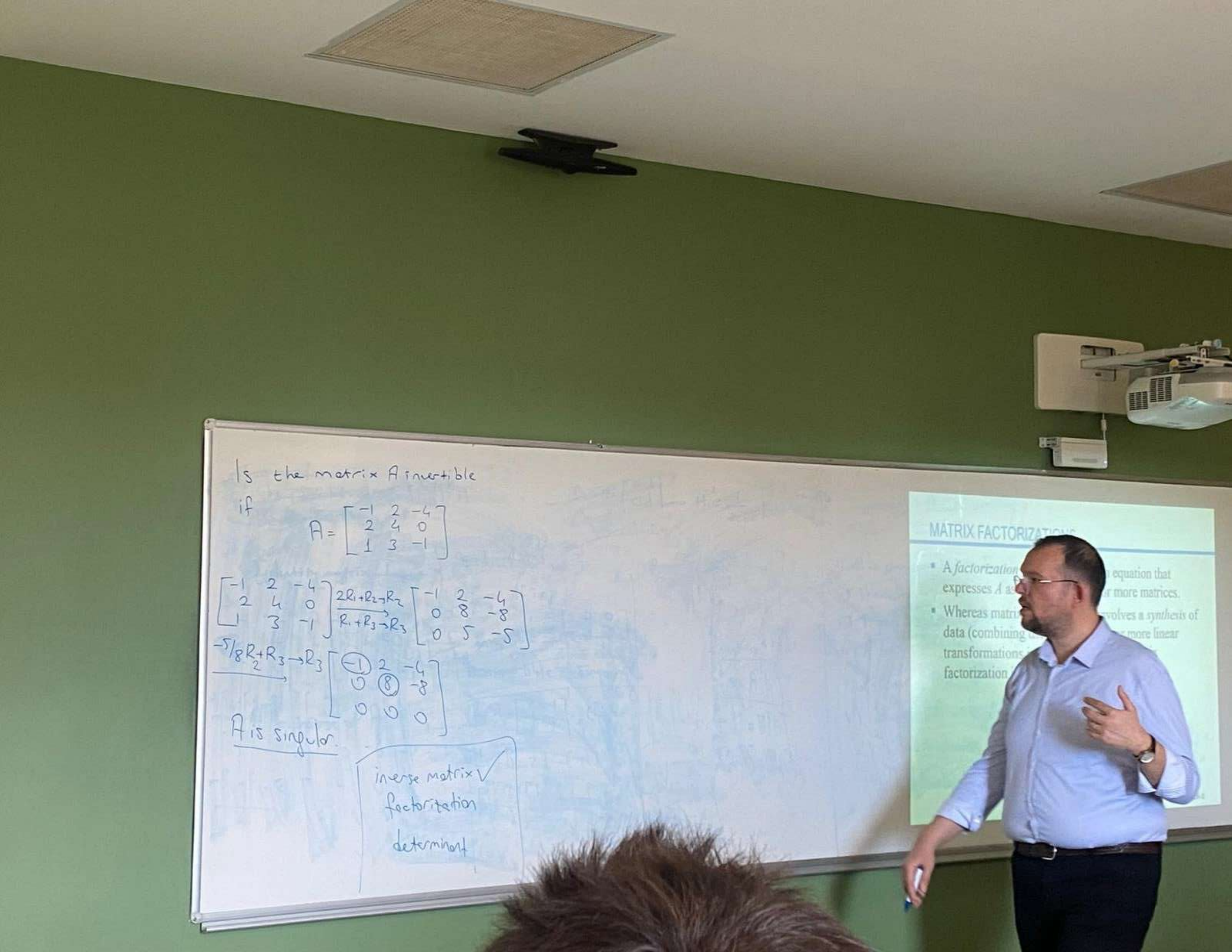

Is the matrix A invertible
if
A = [-1 2 -4; 2 4 0; 1 3 -1]
2R1+R2→R2
R1+R3→R3
-5/8 R2+R3→R3
A is singular.
inverse matrix
factorization
determinant
factorization: A=BC
col1
col2
col3
AN LU FACTORIZATION ALGORITHM
Compare the first columns of A and L. The row operations that create zeros in the first column of A will also create zeros in the first column of L.
To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform A onto U.



Is the matrix A invertible
if
$A = \begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 1 & 3 & -1 \end{bmatrix}$
$\begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 5 & -5 \end{bmatrix}$
$\xrightarrow{-5/8 R_2 + R_3 \to R_3} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$
A is singular
inverse matrix ✓
factoritation
determinant
MATRIX FACTORIZA
A factorization
equation that
expresses A a
or more matrices.
Whereas matri
volves a synthesis of
data (combining
more linear
transformations
factorization

Write the LU factorization for the following metrices

$$A = \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ -6 & -3 & 5 & 1 & 2 \\ -3 & 2 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 3 \\ -6 & -1 & 4 \end{bmatrix} \xrightarrow[-\frac{3}{2}R_1 + R_3 \to R_3]{\frac{1}{2}R_1 + R_2 \to R_2} \begin{bmatrix} -4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\frac{\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}}{-4} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \to 1^{st} \text{ col } L \qquad \xrightarrow{2R_2 + R_3 \to R_3} \underbrace{\begin{bmatrix} -4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 9 \end{bmatrix}}_{U}$$

$$\frac{\begin{bmatrix} 2 \\ -4 \end{bmatrix}}{2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \to 2^{nd} \text{ col of } L.$$

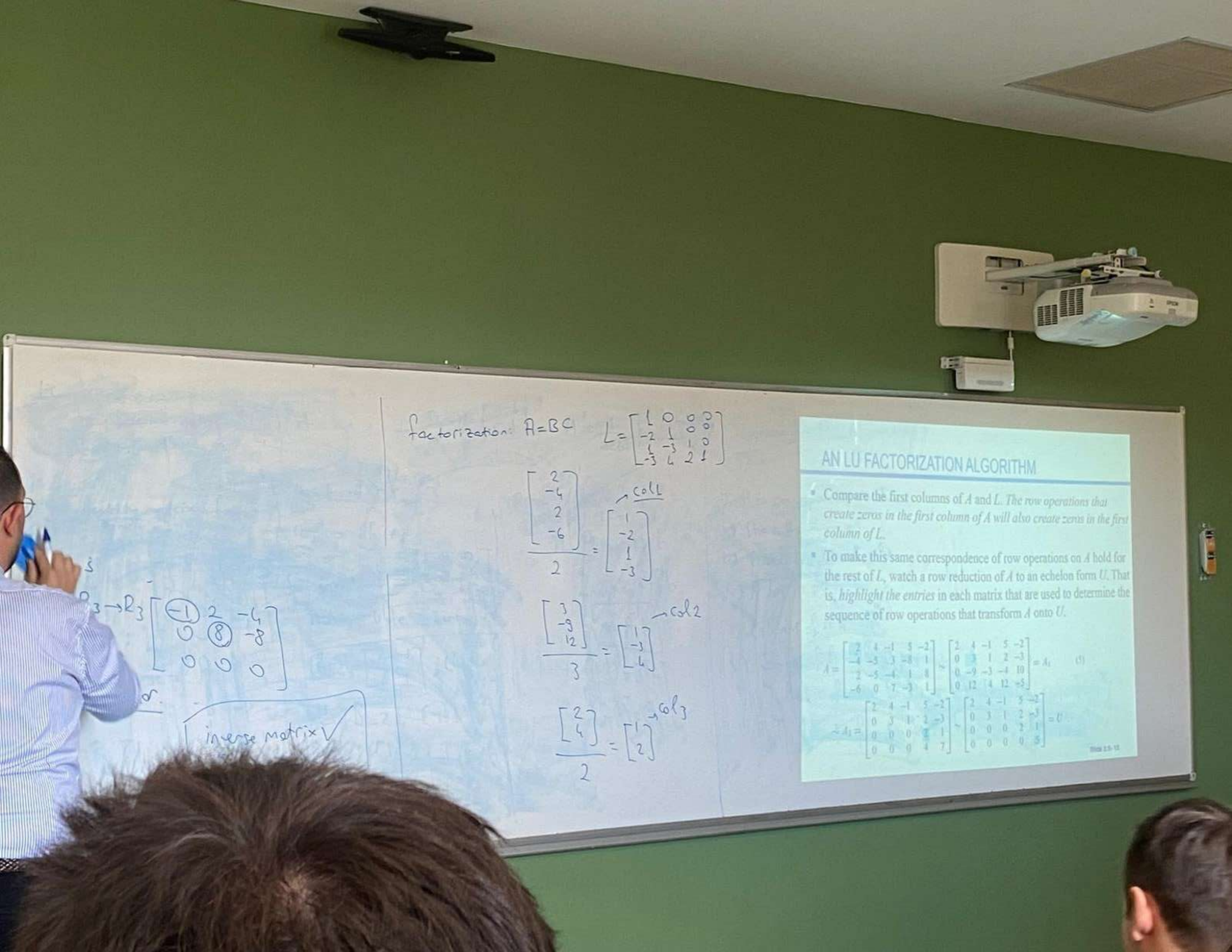
$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -2 & 1 \end{bmatrix}$$

$$A \xrightarrow[R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 1 & 4 & 3 & -1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2 + R_3 \to R_3} \underbrace{\begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 0 & \frac{33}{5} & 4 & -\frac{3}{5} \end{bmatrix}}_{U}$$

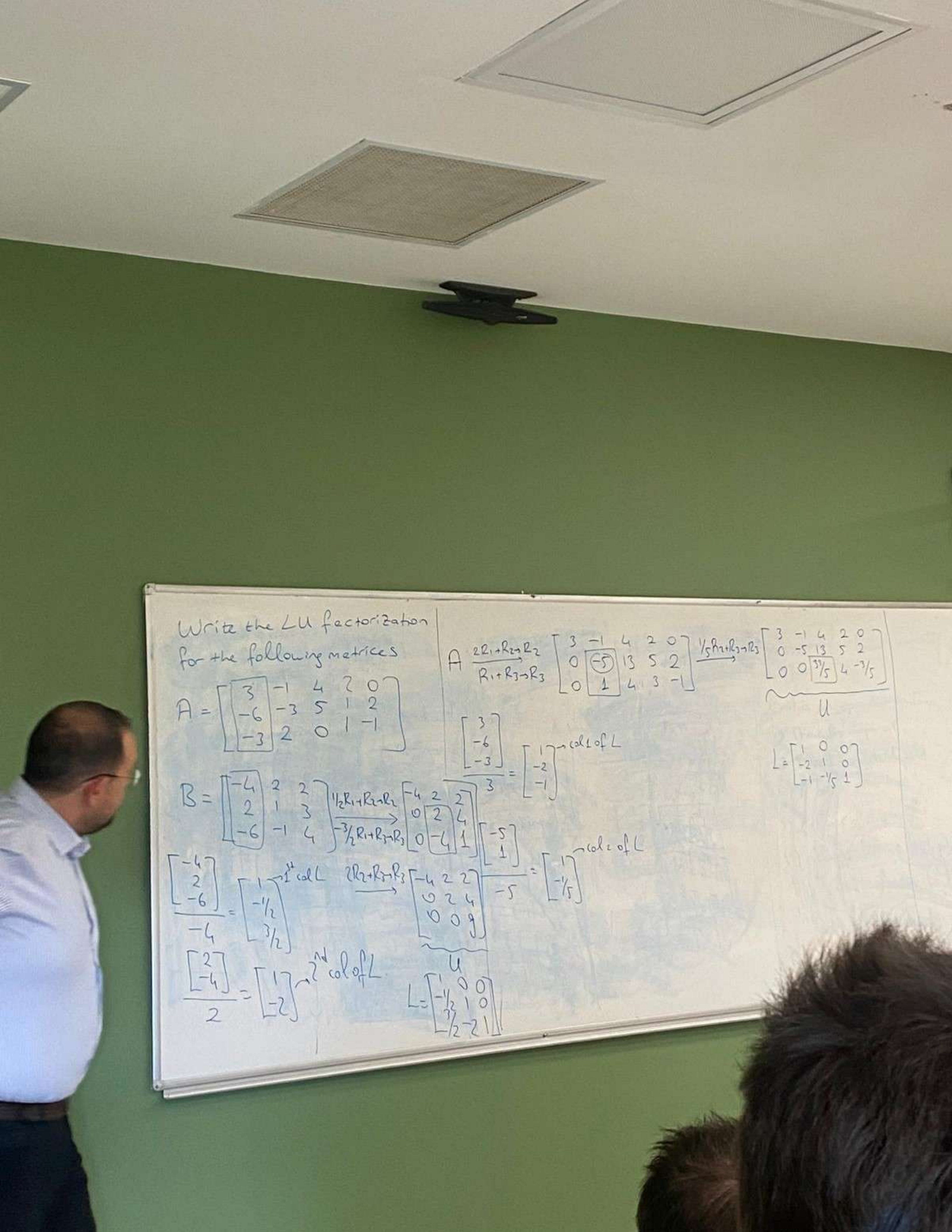
$$\frac{\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}}{3} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \to \text{col 1 of } L$$

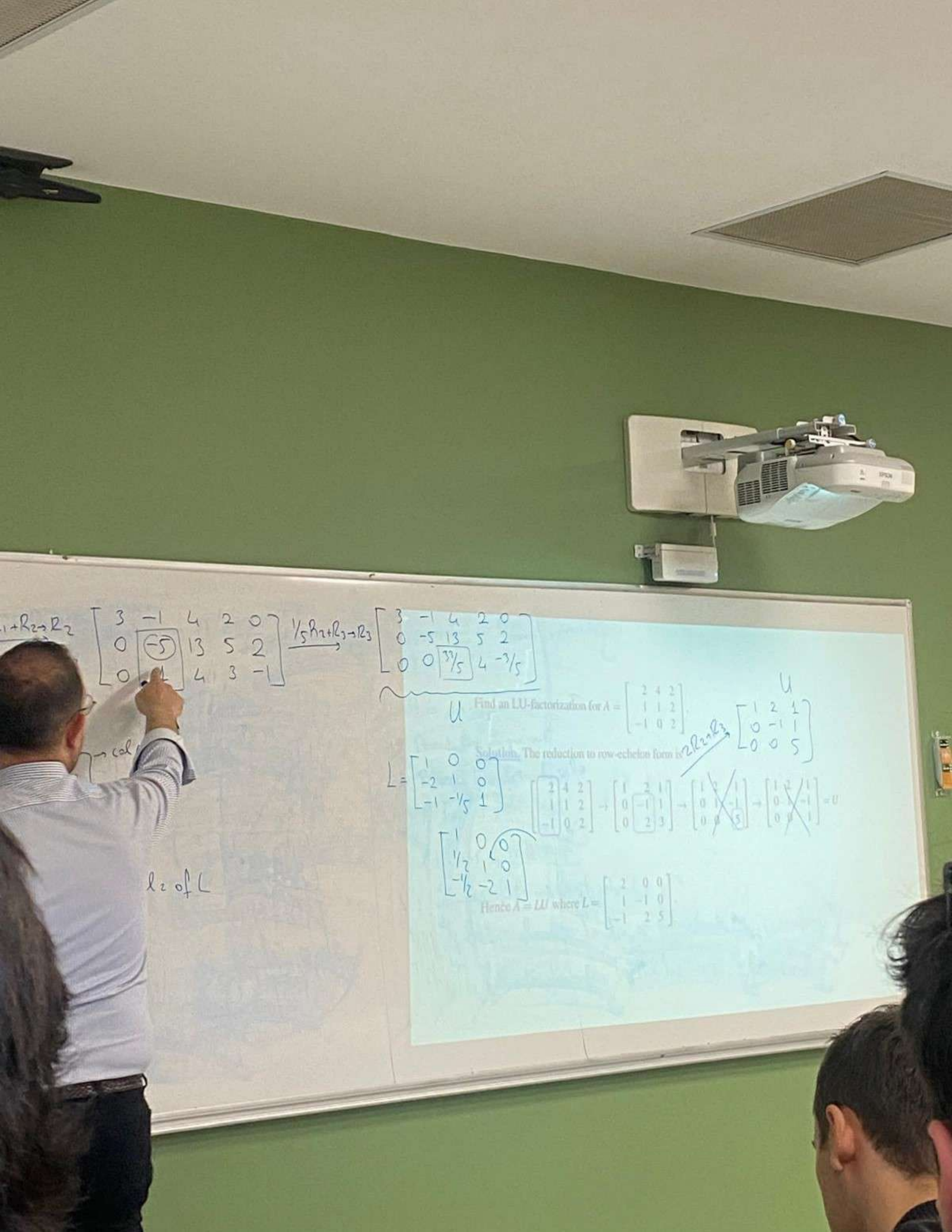
$$\frac{\begin{bmatrix} -5 \\ 1 \end{bmatrix}}{-5} = \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} \to \text{col 2 of } L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -\frac{1}{5} & 1 \end{bmatrix}$$

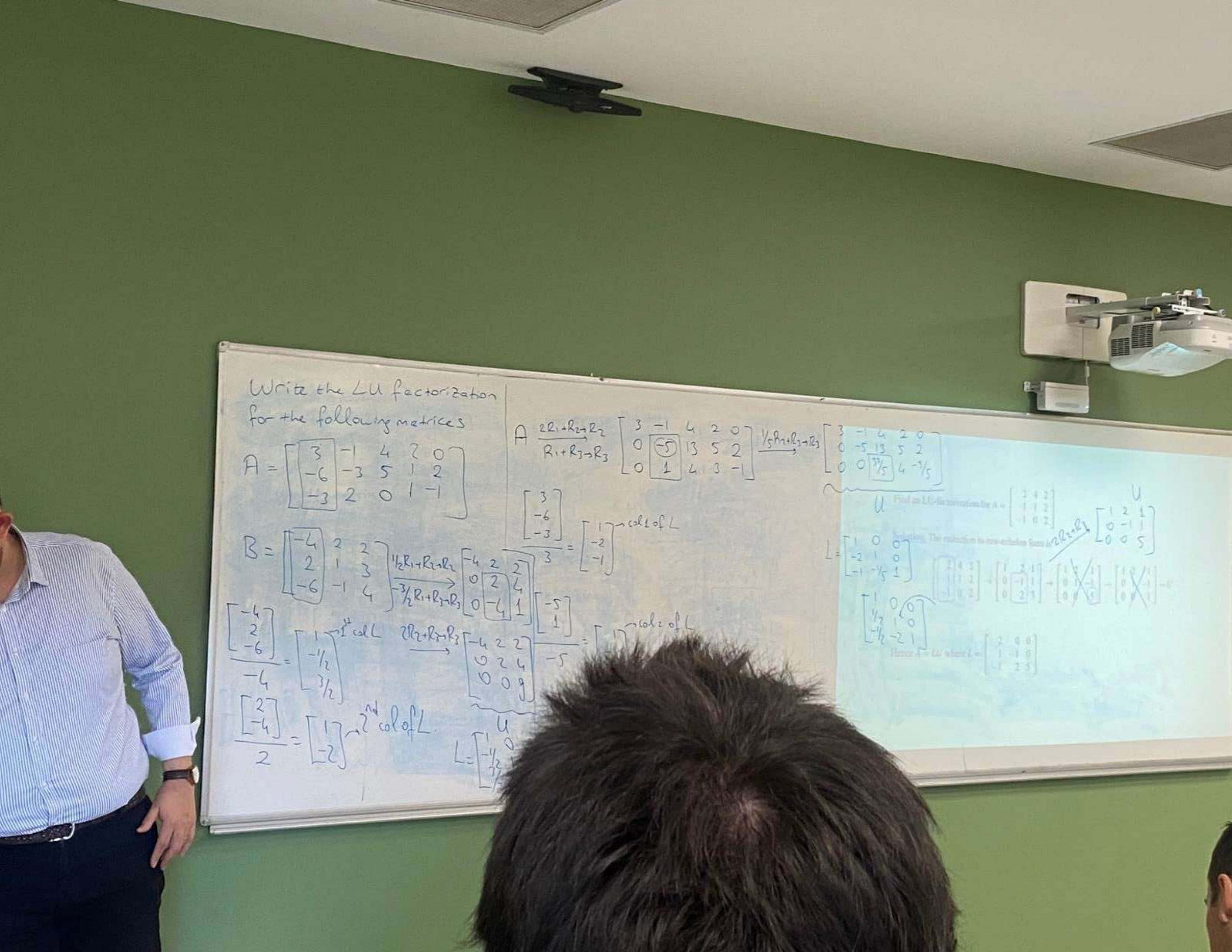


factorization: A=BC
L = [1 0 0 0; -2 1 0 0; 1 -3 1 0; -3 4 2 1]
[2; -4; 2; -6] / 2 = [1; -2; 1; -3] → col1
[3; -9; 12] / 3 = [1; -3; 4] → col2
[2; 4] / 2 = [1; 2] → col3
R3 → R3 [-1 2 -4; 0 8 -8; 0 0 0]
inverse matrix ✓
AN LU FACTORIZATION ALGORITHM
Compare the first columns of A and L. The row operations that create zeros in the first column of A will also create zeros in the first column of L.
To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform A onto U.

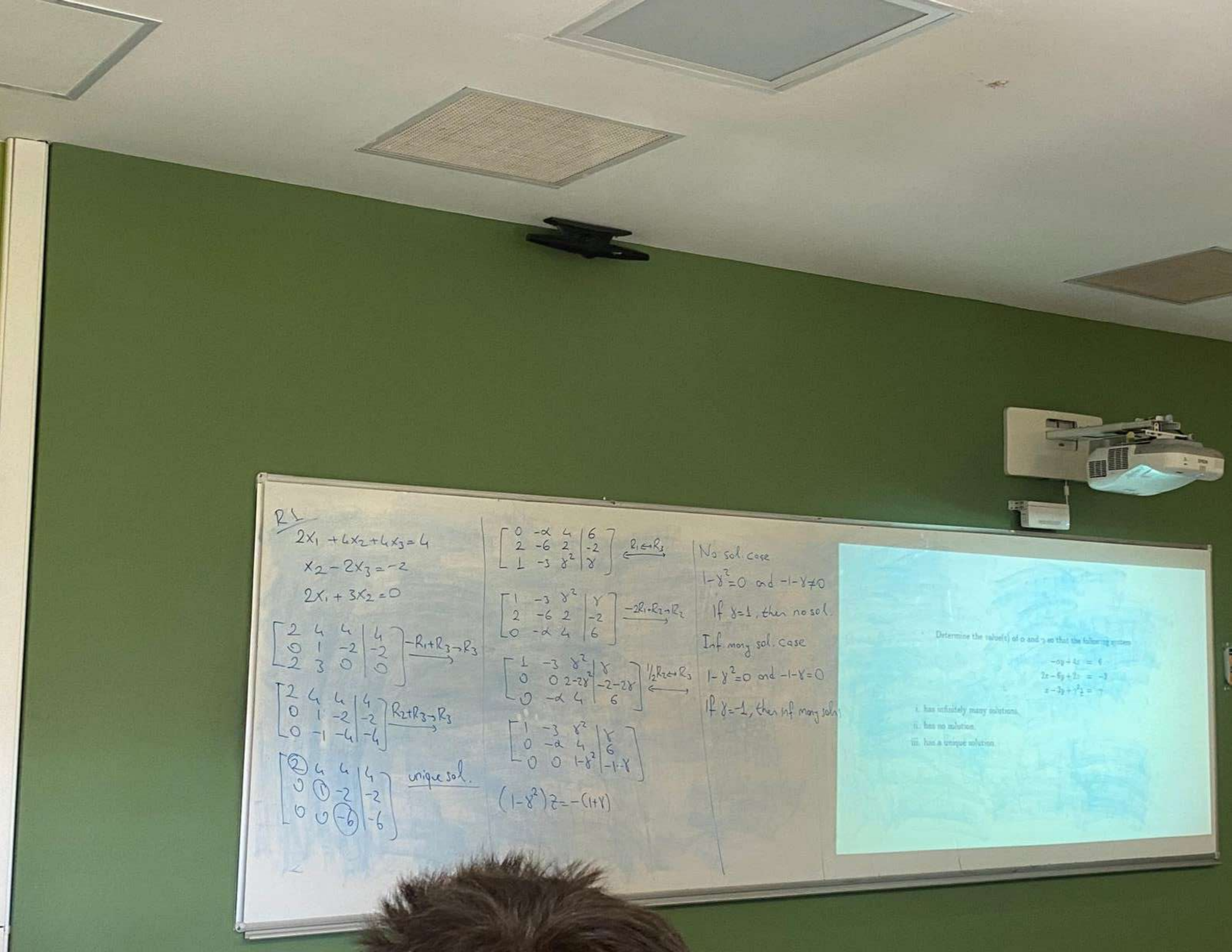

Write the LU factorization for the following matrices
A = [3 -1 4 2 0; -6 -3 5 1 2; -3 2 0 1 -1]
B = [-4 2 2; 2 1 3; -6 -1 4]
1/2 R1 + R2 → R2
-3/2 R1 + R3 → R3
[-4 2 2; 0 2 4; 0 -4 1]
1st col L
2R2 + R3 → R3
[-4 2 2; 0 2 4; 0 0 9]
U
2nd col of L
L = [1 0 0; -1/2 1 0; 3/2 -2 1]
col 2 of L
2R1 + R2 → R2
R1 + R3 → R3
[3 -1 4 2 0; 0 -5 13 5 2; 0 1 4 3 -1]
1/5 R2 + R3 → R3
[3 -1 4 2 0; 0 -5 13 5 2; 0 0 33/5 4 -3/5]
U
col 1 of L
L = [1 0 0; -2 1 0; -1 -1/5 1]

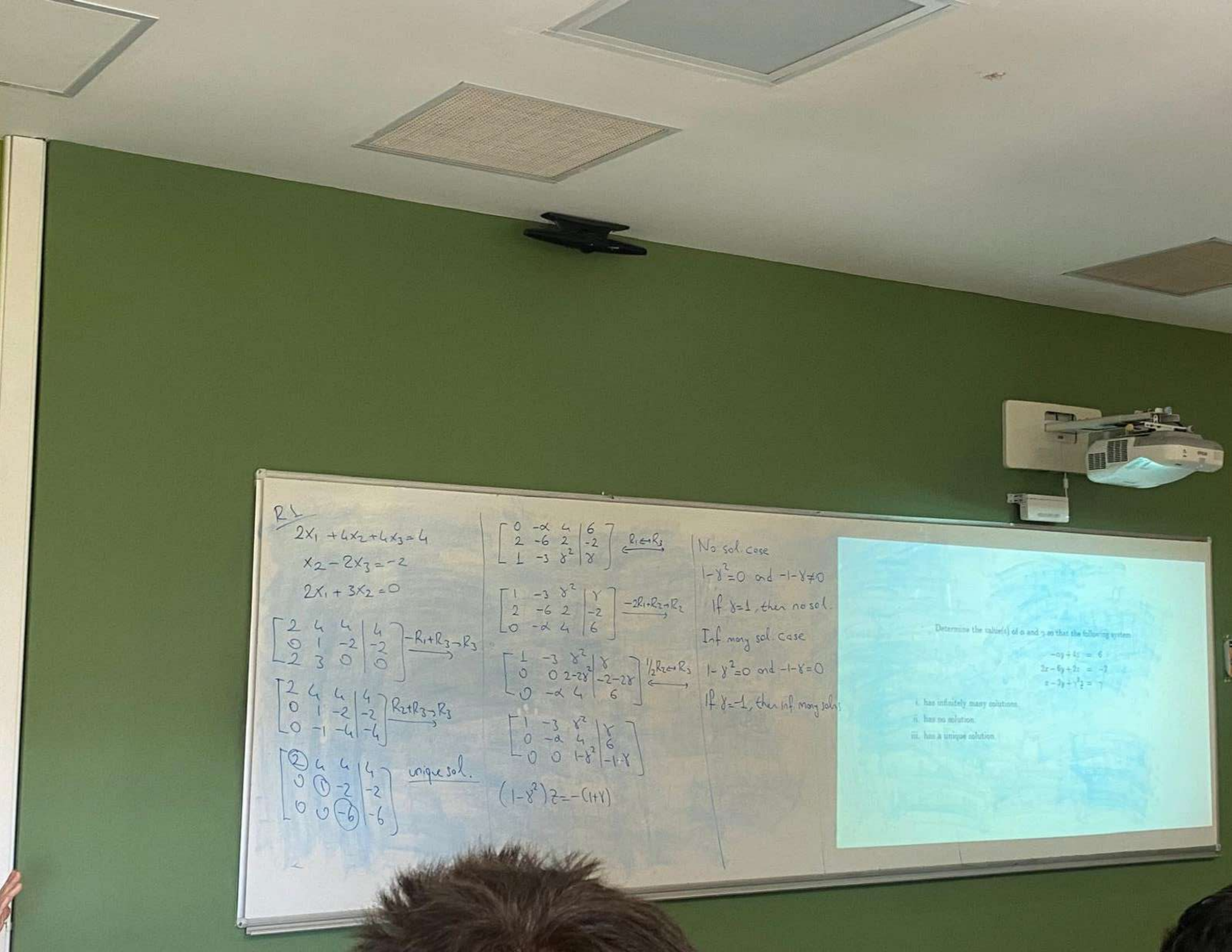


Find an LU-factorization for A =
Solution. The reduction to row-echelon form is
Hence A = LU where L =
U
l₂ of L

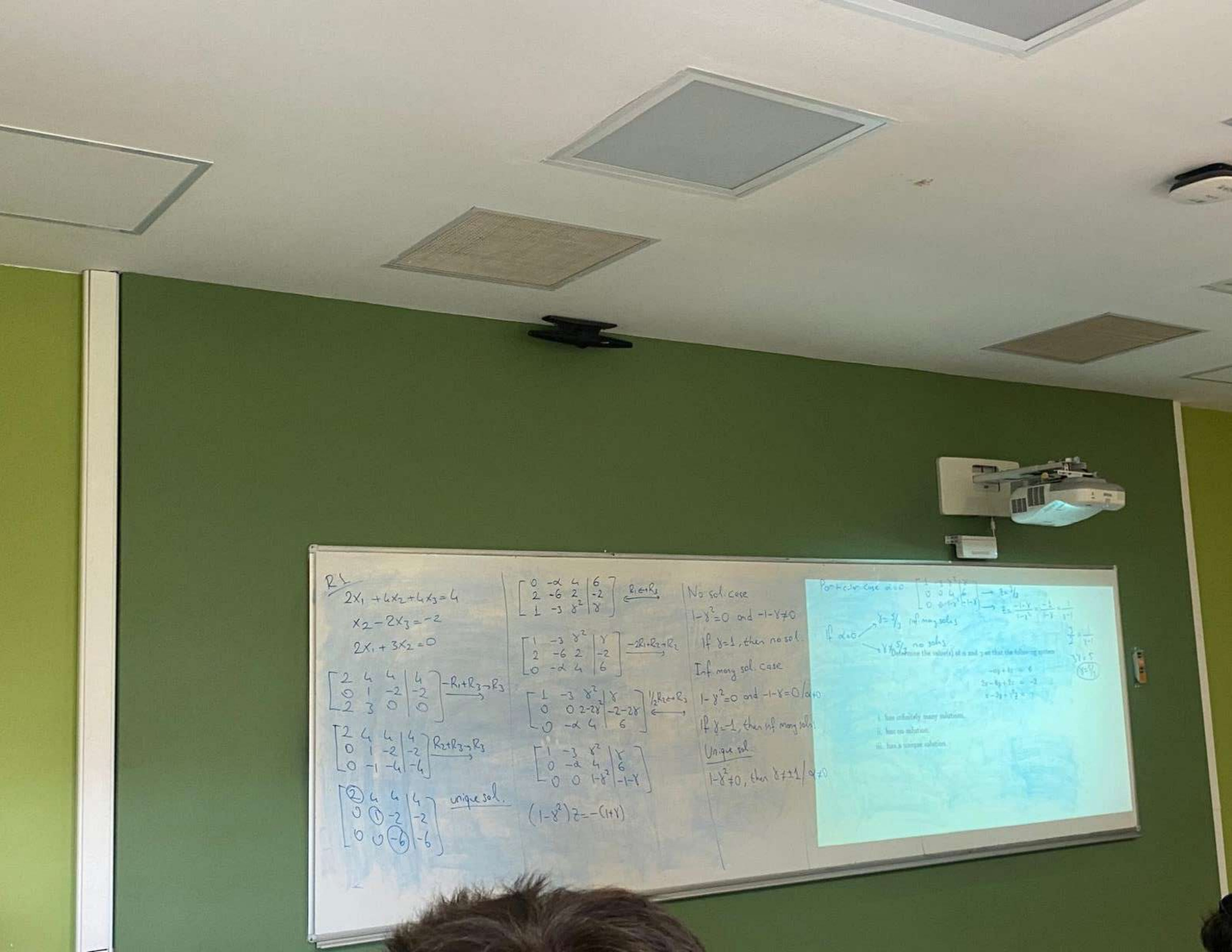


Write the LU factorization
for the following matrices
A =
B =
col 1 of L
1st col L
2nd col of L
col 2 of L
U
L =
Find an LU-factorization for A =
Solution. The reduction to row-echelon form is
Hence A = LU where L =


R1
2x1 + 4x2 + 4x3 = 4
x2 - 2x3 = -2
2x1 + 3x2 = 0
-R1+R3→R3
R2+R3→R3
unique sol.
R1↔R3
-2R1+R2→R2
1/2R2↔R3
(1-γ^2)z = -(1+γ)
No sol. case
1-γ^2=0 and -1-γ≠0
If γ=1, then no sol.
Inf. many sol. case
1-γ^2=0 and -1-γ=0
If γ=-1, then inf many sol.
Determine the value(s) of α and γ so that the following system
i. has infinitely many solutions.
ii. has no solution.
iii. has a unique solution.



R1
$2x_1 + 4x_2 + 4x_3 = 4$
$x_2 - 2x_3 = -2$
$2x_1 + 3x_2 = 0$
$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array}\right] \xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array}\right]$
unique sol.
$\left[\begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{array}\right] \xleftrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{array}\right]$
$(1-\gamma^2)z = -(1+\gamma)$
No sol. case
$1-\gamma^2 = 0$ and $-1-\gamma \neq 0$
If $\gamma = 1$, then no sol.
Inf many sol. case
$1-\gamma^2 = 0$ and $-1-\gamma = 0$
If $\gamma = -1$, then inf many sols
i. has infinitely many solutions.
ii. has no solution.
iii. has a unique solution.


R1
2x1 + 4x2 + 4x3 = 4
x2 − 2x3 = −2
2x1 + 3x2 = 0
−R1+R3→R3
R2+R3→R3
unique sol.
R1↔R3
−2R1+R2→R2
1/2R2↔R3
(1−γ²)z = −(1+γ)
No sol. case
1−γ² = 0 and −1−γ ≠ 0
If γ=1, then no sol.
Inf. many sol. case
1−γ² = 0 and −1−γ = 0 / α≠0
If γ=−1, then inf many sols
Unique sol.
1−γ² ≠ 0, then γ ≠ ±1 / α≠0
Particular case α=0
If α=0
no sols
inf many sols
i. has infinitely many solutions.
ii. has no solution.
iii. has a unique solution.