

$$R^1 \begin{cases} 2x_1 + 4x_2 + 4x_3 = 4 \\ x_2 - 2x_3 = -2 \\ 2x_1 + 3x_2 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_1+R_3 \rightarrow R_3 \\ R_2 \leftrightarrow R_3}}$$

$$\begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{bmatrix} \xrightarrow{R_2+R_3 \rightarrow R_3}$$

$$\begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{bmatrix} \text{ unique sol.}$$

$$\begin{bmatrix} 0 & -x & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & 8^2 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & 8^2 & 8 \\ 2 & -6 & 2 & -2 \\ 0 & -x & 4 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \rightarrow R_2}$$

$$\begin{bmatrix} 1 & -3 & 8^2 & 8 \\ 0 & 0 & 2-2x & -2-2x \\ 0 & -x & 4 & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$(1-x^2)z = -(1+x)$$

No sol case

$$1-x^2=0 \text{ and } -1-x \neq 0$$

If  $x=1$ , then no sol.

If  $x=-1$ , then inf. many sol.

If  $x \neq \pm 1$ , then  $1-x^2 \neq 0$  and  $-1-x \neq 0$

Unique sol.

$$1-x^2 \neq 0, \text{ then } x \neq \pm 1, x \neq -1$$

Particular case case

$$\begin{bmatrix} 0 & 0 & 4 & 6 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

If  $x=1$ , no sol.

If  $x=-1$ , then the number of sol is infinite.

$$\begin{aligned} -4x + 12 &= 8 \\ -4x + 12 &= 8 \\ -4x + 12 &= 8 \\ -4x + 12 &= 8 \end{aligned}$$

$$\begin{bmatrix} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{bmatrix} \xrightarrow{R \leftrightarrow R_1}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{-2R_1 \rightarrow R_2}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \rightarrow R_2}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{bmatrix}$$

$$(1-\gamma^2)z = -(1+\gamma)$$

No sol. case

$$-1-\gamma^2 = 0 \text{ and } -1-\gamma \neq 0$$

If  $\gamma = 1$ , then no sol.

Inf. many sol. case

$$1-\gamma^2 = 0 \text{ and } -1-\gamma = 0 \text{ or } \alpha = 0$$

If  $\gamma = -1$ , then inf. many sol.

Unique sol.

$$-1-\gamma^2 \neq 0, \text{ then } \gamma \neq \pm 1 \text{ or } \alpha \neq 0$$

Is  $\mathbb{R}^3$  spanned by the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

To answer this question

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

If  $A$  is invertible, then  $\mathbb{R}^3$  is spanned.

Answer: yes!



$$\begin{bmatrix} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \delta^2 & \delta \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & \delta^2 & \delta \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_2, -R_2}$$

$$\begin{bmatrix} 1 & -3 & \delta^2 & \delta \\ 0 & 0 & 2-2\delta^2 & -2-2\delta \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & \delta^2 & \delta \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\delta^2 & -1-\delta \end{bmatrix}$$

$$(1-\delta^2)z = -(1+\delta)$$

No sol case

$$1-\delta^2 = 0 \text{ and } -1-\delta \neq 0$$

If  $\delta = 1$ , then no sol.

If many sol. case

$$1-\delta^2 = 0 \text{ and } -1-\delta = 0 \text{ or } \alpha \neq 0$$

If  $\delta = -1$ , then inf many sol.

Unique sol.

$$1-\delta^2 \neq 0, \text{ then } \delta \neq \pm 1, \alpha \neq 0$$

Is  $\mathbb{R}^3$  spanned by the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$$

To answer this question

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

If  $A$  is invertible, then  $\mathbb{R}^3$  is spanned.  
(Checking...)



$$\begin{bmatrix} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & \gamma^2 & \gamma \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_2}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 0 & 0 & 2-2\gamma^2 & -2-2\gamma \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & \gamma^2 & \gamma \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\gamma^2 & -1-\gamma \end{bmatrix}$$

$(1-\gamma^2)z = -(1+\gamma)$   
 No sol. case  
 $1-\gamma^2 = 0$  and  $-1-\gamma \neq 0$   
 If  $\gamma = 1$ , then no sol.  
 Inf many sol. case  
 $1-\gamma^2 = 0$  and  $-1-\gamma = 0$  or  $\alpha = 0$   
 If  $\gamma = -1$ , then inf many sol.  
 Unique sol  
 $1-\gamma^2 \neq 0$ , then  $\gamma \neq \pm 1$  or  $\alpha \neq 0$

Let  $S$  be a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

Write the kernel of  $S$

Kernel

To answer this question

$S(x, y, z) = (x - 3y + 2z, 3x + 2y - 2z, 2x - 3y + 2z)$

$A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & -2 \\ 2 & -3 & 2 \end{bmatrix}$

If  $A$  is invertible, then  $\text{ker } S = \{0\}$

Otherwise, no!

$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & -2 \\ 2 & -3 & 2 \end{bmatrix}$



$$\left[ \begin{array}{c} R_1 \leftrightarrow R_3 \\ \leftarrow \end{array} \right]$$

$$\left[ \begin{array}{c} -2R_1 + R_2 \rightarrow R_2 \\ \leftarrow \end{array} \right]$$

$$\left[ \begin{array}{c} \frac{1}{2}R_2 \leftrightarrow R_3 \\ \leftarrow \end{array} \right]$$

$$\left[ \begin{array}{c} \frac{1}{6} \\ -1-x \end{array} \right]$$

No sol. case

$$1-x^2=0 \text{ and } -1-x \neq 0$$

If  $x=1$ , then no sol.

Inf many sol. case

$$1-x^2=0 \text{ and } -1-x=0/x \neq 0$$

If  $x=-1$ , then inf many sol.

Unique sol.

$$1-x^2 \neq 0, \text{ then } x \neq \pm 1/x \neq 0$$

Singular

Let  $S$  be a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

$$S(x_1, x_2, x_3) = (x_1 - x_3 + 2x_2, 2x_1 + x_2 - x_1 - x_2 + 2x_3)$$

Write the kernel of  $S$ .

$$x_1 - x_2 + 2x_3 = 0$$

$$2x_1 + x_2 = 0$$

$$-x_1 - x_2 + 2x_3 = 0$$

We need to solve  $Ax=0$  if

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

The augmented matrix

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_1+R_2, R_1+R_3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -4 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{2R_2+R_3}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 3 & -4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{array} \right]$$

To answer this question

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

If  $A$  is invertible, answer is Yes!

Otherwise, no!

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\{x=0\}$  is 1-1 and onto

$$\left[ \begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -5 & 8^2 & 8 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[ \begin{array}{ccc|c} 1 & -5 & 8^2 & 8 \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array} \right] \xrightarrow{\substack{-2R_1 \leftrightarrow R_2 \\ -8R_1 \leftrightarrow R_3}}$$

$$\left[ \begin{array}{ccc|c} 1 & -5 & 8^2 & 8 \\ 0 & 2-2\alpha & -14 & -18 \\ 0 & -\alpha & 4 & 6 \end{array} \right] \xrightarrow{\substack{1/2 R_2 \leftrightarrow R_2 \\ 1/2 R_3 \leftrightarrow R_3}}$$

$$\left[ \begin{array}{ccc|c} 1 & -5 & 8^2 & 8 \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-8^2 & -14 \end{array} \right]$$

$$(1-8^2)z = -(14y)$$

No sol. case

$$1-8^2 = 0 \text{ and } -1-8 \neq 0$$

If  $8=1$ , then no sol.

Inf many sol. case

$$1-8^2 = 0 \text{ and } -1-8 = 0 \text{ or } \alpha = 0$$

If  $8 = -1$ , then inf many sol.

Unique sol.

$$1-8^2 \neq 0, \text{ then } 8 \neq \pm 1, \alpha \neq 0$$

Let S be a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

$$S(x, y, z) = (\alpha x - \alpha y + 2\alpha z, 2\alpha x + \alpha y - \alpha z, -\alpha x - \alpha y + \alpha z)$$

Write the kernel of S

Singular

To answer this question

$$A = \begin{bmatrix} \alpha & -\alpha & 2\alpha \\ 2\alpha & \alpha & -\alpha \\ -\alpha & -\alpha & \alpha \end{bmatrix} \xrightarrow{\substack{-R_1 \leftrightarrow R_2 \\ -R_1 \leftrightarrow R_3}} \begin{bmatrix} \alpha & -\alpha & 2\alpha \\ 0 & 3\alpha & -3\alpha \\ 0 & \alpha & -\alpha \end{bmatrix} \xrightarrow{\substack{1/3 R_2 \leftrightarrow R_2 \\ -R_2 \leftrightarrow R_3}} \begin{bmatrix} \alpha & -\alpha & 2\alpha \\ 0 & 3\alpha & -3\alpha \\ 0 & 0 & 0 \end{bmatrix}$$

If  $\alpha$  is nonzero, solve a  $2 \times 2$  system. Otherwise, no!

$$\begin{bmatrix} \alpha & -\alpha & 2\alpha \\ 0 & 3\alpha & -3\alpha \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Check the invertibility of the matrix

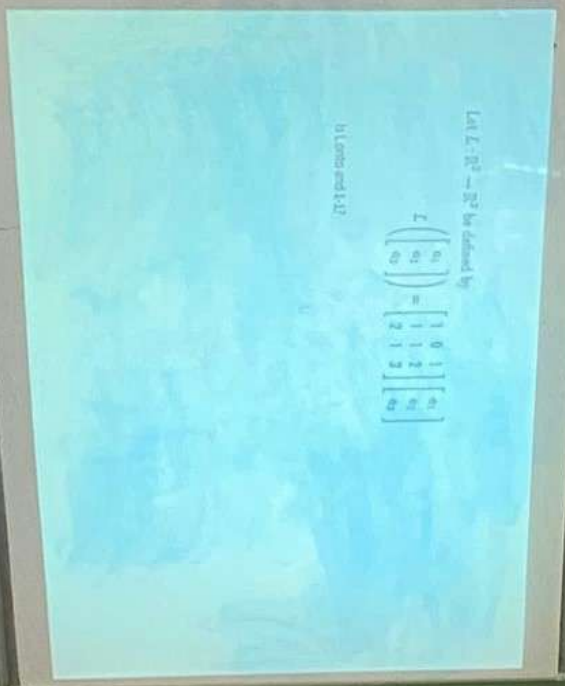
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{-R_1+R_2 \\ -2R_1+R_3}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A has no inverse

$AX=0$  has non trivial sol. (T is not HI)

Columns of A do not span  $\mathbb{R}^3$  (T is not onto)



## Determinant

- Minor of matrix
- Expansion for det

Def: Suppose  $A$  is an  $n \times n$  matrix. A minor of  $A$ , denoted by  $M_{ij}$ , is the  $(n-1) \times (n-1)$  matrix which is obtained by deleting the  $i$ th row and  $j$ th col of the matrix  $A$ .

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & -2 & 1 & 4 \\ 1 & 1 & 0 & 2 \\ -3 & -3 & -1 & -1 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Write  $M_{23}$  for  $A$ , and  $M_{31}$  for  $B$

$$M_{23}^A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M_{31}^B = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Def: (Cofactor) A cofactor of an  $n \times n$  matrix  $A$ , denoted by  $C_{ij}$ , is a real number which is obtained by

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

△ Determinant is an iterative process and we should keep writing minor in this procedure until they become  $2 \times 2$

For example, let us write all cofactors of

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \det(M_{11}) = \begin{vmatrix} 1 & 3 \\ -3 & 0 \end{vmatrix} = 9$$

$$C_{12} = (-1)^{1+2} \det(M_{12}) = - \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$C_{13} = (-1)^{1+3} \det(M_{13}) = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$C_{21} = (-1)^{2+1} \det(M_{21}) = - \begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2} \det(M_{22}) = \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3} \det(M_{33}) = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$



Write  $M_{23}$  for A, and  $M_{31}$  for B.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Def (Cofactor) A cofactor of an  $n \times n$  matrix A, denoted by  $C_{ij}$ , is a real number which is obtained by

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

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$$C_{13} = (-1)^{1+3} \det(M_{13}) = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$C_{21} = (-1)^{2+1} \det(M_{21}) = - \begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2} \det(M_{22}) = \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{23} = (-1)^{2+3} \det(M_{23}) = - \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = 8$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = -6$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

$$\begin{bmatrix} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & 8^2 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & 8^2 & 8 \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \rightarrow R_2}$$

$$\begin{bmatrix} 1 & -3 & 8^2 & 8 \\ 0 & 0 & 2-2\alpha^2 & -2-2\alpha \\ 0 & -\alpha & 4 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -3 & 8^2 & 8 \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-\alpha^2 & -1-\alpha \end{bmatrix}$$

$$(1-\alpha^2)$$

No sol. case

$$1-\alpha^2 = 0 \text{ and } -1-\alpha \neq 0$$

If  $\alpha = 1$ , then no sol.

Inf. many sol. case

$$1-\alpha^2 = 0 \text{ and } -1-\alpha = 0 / \alpha \neq 0$$

If  $\alpha = -1$ , then inf. many sol.

Unique sol.

$$1-\alpha^2 \neq 0, \text{ then } \alpha \neq \pm 1 / \alpha \neq 0$$

Singular

Let  $S$  be a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  defined by

$$S(x_1, x_2) = (2x_1 - 3x_2 + 2x_3, 2x_1 - 2x_2 + 4x_3)$$

Write the kernel of  $S$

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

We need to solve  $Ax = 0$  if

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

The augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{-2R_1 + R_2, -R_1 + R_3}$$

$$\frac{2}{3}R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

To answer this question

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

If  $A$  is invertible, answer is Yes!

Otherwise, no!

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_3}$$

$$\ker S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

$S$  is 1-1 and onto

## Determinant

Def (Cofactor expansion) Let  $A$  be an  $n \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

$i^{\text{th}}$  row  $\leftarrow$   $j^{\text{th}}$  col  $\rightarrow$

While evaluating the determinant, it is enough to pick one row or one column of a matrix.

$i^{\text{th}}$  row

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

$$= \sum_{k=1}^n a_{ik}C_{ik}$$

$j^{\text{th}}$  col:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$

$$= \sum_{k=1}^n a_{kj}C_{kj}$$

for example:  
Cofactors of

$$C_{11} = (-1)^{1+1} \det A_{11}$$

$$C_{12} = (-1)^{1+2} \det A_{12}$$

$$C_{13} = (-1)^{1+3} \det A_{13}$$

$$C_{21} = (-1)^{2+1} \det A_{21}$$

$$C_{22} = (-1)^{2+2} \det A_{22}$$

$$+a_{12}c_{12} + \dots + a_{1n}c_{1n}$$

Ex: Evaluate the following determinants

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \det A = 1c_{31} + 0c_{32} + 1c_{33}$$
$$= (-1)^4 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + (-1)^6 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = -2 + 7 = 5$$

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 4 & 1 \\ -1 & 2 & -2 \end{bmatrix} \rightarrow \det B = -1c_{11} + 0c_{21} - 1c_{31}$$
$$= -(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - (-1)^4 \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix}$$
$$= 10 - 10 = 0$$

$$C = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 3 & 1 \\ 4 & 1 & 0 & -2 \end{bmatrix}$$

$$+ a_{1j}c_{1j}$$



Ex: Evaluate the following determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

+  $a_{in}c_{in}$

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \det A = 1C_{31} + 0C_{32} + 1C_{33}$$

$$= (-1)^4 \begin{vmatrix} -1 & 4 \\ 5 & 1 \end{vmatrix} + (-1)^6 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = -2 + 7 = 5$$

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 4 & 1 \\ -1 & 2 & -2 \end{bmatrix}$$

$$\rightarrow \det B = -1C_{11} + 0C_{21} - 1C_{31}$$

$$= -(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - (-1)^4 \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= 10 - 10 = 0$$

+  $a_{nj}c_{nj}$

$$C = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 3 & 1 \\ 4 & 1 & 0 & -2 \end{bmatrix}$$

$$\det C = 3C_{33} = 3(-1)^6 \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= 3 \left[ 2(-1)^3 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \right]$$

$$= 3[-10 - 2 - 22] = -102$$

Write  $M_{23}$  for A, and  $M_{31}$  for B.

$$M_{23}^A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M_{31}^B = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Def (Cofactor) A cofactor of an  $n \times n$  matrix A, denoted by  $C_{ij}$ , is a real number which is obtained

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

$\triangle$  Determinant is an iterative process and we should keep writing minor in this procedure until they become  $2 \times 2$

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$$C_{12} = (-1)^{1+2} \det(M_{12}) = - \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 6$$

$$C_{13} = (-1)^{1+3} \det(M_{13}) = \begin{vmatrix} 0 & 1 \\ 2 & -3 \end{vmatrix} = -2$$

$$C_{21} = (-1)^{2+1} \det(M_{21}) = - \begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2} \det(M_{22}) = \begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{23} = (-1)^{2+3} \det(M_{23}) = - \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = 8$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = -6$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

① If  $A_{n \times n}$  is triangular, then  $\det A = \prod_{i=1}^n a_{ii} = \underbrace{5 \cdot 4 \cdot 2 \cdot 1}_{\det A}$

② - Replacement has no effect on  $\det$

- Interchanging  $\det B = -\det A$

- Scaling  $\det B = k \det A$

- If  $B_{n \times n} = k A_{n \times n}$ , then  $\det B = k^n \det A$

For example evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 0 \\ 1 & -1 & 4 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix} \xrightarrow{\substack{R_2+R_1 \\ R_3-R_1 \\ R_4-2R_1}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 5 & -4 & -3 \end{bmatrix}$$

### 1.2 PROPERTIES OF DETERMINANTS

- Property 1: Let  $A$  be a square matrix
  - If a multiple of one row of  $A$  is added to another row to produce a matrix  $B$ , then  $\det B = \det A$ .
  - If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
  - If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \det A$ .

① If  $A_{nn}$  is triangular, then  $\det A = \prod_{i=1}^n a_{ii}$

② Replacement has no effect on det.

- interchanging  $\det B = -\det A$

- Scaling  $\det B = k \det A$

- If  $B_{nn} = k A_{nn}$ , then  $\det B = k^n \det A$

For example evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 0 \\ 2 & -1 & 4 & -1 \\ 3 & 0 & 0 & -1 \end{bmatrix}$$

$\begin{matrix} R_1+R_2 \\ R_1+R_3 \\ R_1+R_4 \end{matrix}$

$\det = -16$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -4 & -3 \end{bmatrix}$$

$\begin{matrix} R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 + R_2 \end{matrix}$

$\det = -16$

$-5R_2+R_3+R_4$

$-1/2 R_3$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$\begin{matrix} R_1+R_2 \\ R_4 \rightarrow R_4 + 8R_3 \end{matrix}$

$\det = -16$

$\det = -16$

**3.2 PROPERTIES OF DETERMINANTS**

- Property 3: Let  $A$  be a square matrix
  - a) If a multiple of one row of  $A$  is added to another row to produce a matrix  $B$ , then  $\det B = \det A$ .
  - b) If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
  - c) If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \cdot \det A$ .



$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix} \det = -16$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

Det = -16

### 3.2 PROPERTIES OF DETERMINANTS

- Property 3: Let  $A$  be a square matrix
  - a) If a multiple of one row of  $A$  is added to another row to produce a matrix  $B$ , then  $\det B = \det A$ .
  - b) If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
  - c) If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \cdot \det A$ .

① If  $A_{n \times n}$  is triangular, then  $\det A = \prod_{i=1}^n a_{ii}$

② Replacement has no effect on det.

- interchanging  $\det B = -\det A$

- scaling  $\det B = k \det A$

- If  $B_{n \times n} = k A_{n \times n}$ , then  $\det B = k^n \det A$

for example evaluate the determinant of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 0 \\ 1 & -1 & 4 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix} \xrightarrow{\substack{R_1+R_2 \rightarrow R_2 \\ -R_1+R_3 \rightarrow R_3 \\ -2R_1+R_4 \rightarrow R_4}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 5 & -4 & -3 \end{bmatrix} \det = -16$$

$\det = -16$

$$\xrightarrow{-5R_2+R_4 \rightarrow R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -8 \end{bmatrix} \det = -16$$

$$\xrightarrow{-\frac{1}{2}R_3+R_4 \rightarrow R_4} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix} \det = -16$$

Ex: Evaluate the following determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \det A = 1C_{31} + 0C_{32} + 1C_{33}$$

$$= (-1)^4 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + (-1)^6 \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = -2 + 7 = 5$$

$$B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 4 & 1 \\ -1 & 2 & -2 \end{bmatrix} \rightarrow \det B = -1C_{11} + 0C_{21} - 1C_{31}$$

$$= -(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - (-1)^4 \begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= 10 - 10 = 0$$

$$C = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 1 & 3 & 0 & -1 \\ 0 & 2 & 3 & 1 \\ 4 & 1 & 0 & -2 \end{bmatrix}$$

$$\det C = 3C_{33} = 3(-1)^6 \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= 3 \left[ 2(-1)^3 \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} \right]$$

$$= 3[-10 - 2 - 22] = -62$$

Ex. Consider

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = W$$

Find

$$\begin{vmatrix} 2c & 2f & 2i \\ b+a & d+e & g+h \\ -a & -d & -g \end{vmatrix}$$

$(A + \det B)$

$$\begin{array}{l} \begin{array}{c} \textcircled{W} \\ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{T} \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \xrightarrow{r_1+r_2 \rightarrow r_1} \begin{vmatrix} a & d & g \\ b+a & e+d & g+h \\ c & f & i \end{vmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{vmatrix} c & f & i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix} \end{array} \\ \begin{array}{c} \textcircled{W} \\ \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix} \xrightarrow{2R_1} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix} \xrightarrow{-R_3} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix} \end{array} \\ \begin{array}{c} \textcircled{-W} \\ \begin{vmatrix} c & f & i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix} \end{array} \\ \begin{array}{c} \textcircled{-2W} \\ \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix} \end{array} \\ \begin{array}{c} \textcircled{2W} \\ \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix} \end{array} \end{array}$$

✓Cramer's rule

Adjoint of a matrix.

Defn: (Cramer's rule) This rule is used in the solutions of linear systems for which we are interested in only one particular unknown.

Suppose that  $A$  is an  $n \times n$  matrix which is expressed in terms of its columns so that

$$A = [C_1 \ C_2 \ \dots \ C_i \ \dots \ C_n]$$

Consider the system

$$Ax = b$$

If we are interested in only the  $i^{\text{th}}$  entry of  $\vec{x}$ , namely  $x_i$ , we define a new matrix  $A_i$  by replacing the  $i^{\text{th}}$  column of  $A$  with vector  $b$ , that is

$$A_i = [C_1 \ C_2 \ \dots \ C_{i-1} \ b \ C_{i+1} \ \dots \ C_n]$$

$$x_i = \frac{\det A_i}{\det A}$$

Consider the system  
 $Ax = b$

If we are interested in only the  
 $i^{\text{th}}$  entry of  $\vec{x}$ , namely  $x_i$ ,  
 we define a new matrix  $A_i$  by  
 replacing the  $i^{\text{th}}$  column of  $A$   
 with vector  $b$ , that is

$$A_i = [C_1 \ C_2 \ \dots \ C_{i-1} \ b \ C_{i+1} \ \dots \ C_n]$$

$$x_i = \frac{\det A_i}{\det A}$$

Ex 1 For the given linear system

$$7x - y + 2z = 1$$

$$2x + 3y - z = 2$$

$$3x - 2y + z = -1$$

find only  $y$  without having  $x$  and  $z$

$$A = \begin{bmatrix} 7 & -1 & 2 \\ 2 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{y det } A_i = 1$$

$$\det A = 7C_{11} - C_{12} + 2C_{13}$$

$$= 7(-1)^1 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} - (-1)^1 \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}$$

$$= 7 + 5 - 26 = -14 \neq 0$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}$$

$$\det A_2 = 2C_{11} - C_{12} + C_{13}$$

$$= 2(-1)^1 \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix}$$

$$- (-1)^1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$+ (-1)^1 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

$$= -16 - 10 + 12$$

$$= -14$$

Ex 2: Find only  $z$  for the system

$$\begin{aligned} -x + 2y - z &= 4 \\ 2x + 5y + 3z &= 2 \\ -y - 2z &= -3 \end{aligned}$$

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & 5 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\det A = 0C_{31} - C_{32} - 2C_{33}$$

$$= -(-1)^5 \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} - 2(-1)^6 \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= -1 + 18 = 17 \neq 0$$

Define

$$\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$$

$$\det A_3 = \det \begin{bmatrix} -1 & 2 & -1 \\ 2 & 5 & 3 \\ 0 & -1 & -2 \end{bmatrix} = 2C_{21}$$

$$= 2 \begin{vmatrix} 5 & 2 \\ -1 & -3 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix}$$

$$= 17$$

$$z = \frac{\det A_3}{\det A} = 1$$

Ex 1: For the given linear system

$$\begin{aligned} 2x - y + 2z &= 7 \\ 2x + 3y - z &= 2 \\ 3x - 2y + z &= 1 \end{aligned}$$

find only  $y$  without  $x$  and  $z$

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\det A = 7C_{11} - C_{12} + 2C_{13}$$

$$= 7(-1)^2 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} -$$

$$= 7 + 5 - 26 = -14$$

Ex. Consider

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = W$$

Find

$$\begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{T} \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \xrightarrow{R_1+R_2 \rightarrow R_2} \begin{vmatrix} a & d & g \\ b+a & e+d & g+h \\ c & f & i \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} c & f & i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix}$$
  
$$\xrightarrow{2R_1} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ a & d & g \end{vmatrix} \xrightarrow{-R_3} \begin{vmatrix} 2c & 2f & 2i \\ b+a & e+d & g+h \\ -a & -d & -g \end{vmatrix}$$

$\det(A + \det B)$





Define  $A_3 = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 5 & 2 \\ 3 & -1 & 3 \end{bmatrix}$

$$\begin{aligned} \det A_3 &= - \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} + 4 \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} \\ &= -1(-1) - 2(-1) + 4(-13) \\ &= 13 \end{aligned}$$

$$z = \frac{\det A_3}{\det A} = 1.$$

Ex Solve the

he

D.P.

$$A_1 = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 5 & 2 \\ 0 & -1 & -3 \end{bmatrix}$$

$$-1 C_{11} + 2 C_{21}$$

$$-1 (-1)^2 \begin{vmatrix} 5 & 2 \\ -1 & -3 \end{vmatrix} + 2 (-1)^3 \begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix}$$

$$= 13 + 6 = 17$$

$$t = \frac{\text{adj} A}{\det A} = 1$$

$$\begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix}$$

Linear Independence

Suppose that  $\{v_1, v_2, \dots, v_n\}$  be a set of vectors. Then the given set is said to be a linearly independent set (the vectors are called linearly independent)

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$   
 holds for  $c_1 = c_2 = \dots = c_n = 0$

Remark 1. In  $\mathbb{R}^n$ , we can write at most a linearly independent vectors. Any  $n+1$  vectors are linearly dependent.

Remark 2. Suppose that the set  $\{v_1, \dots, v_n\}$  is linearly dependent. Then, there is at least one vector  $v_i$  in this set so that

$v_i \in \text{span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$

Remark 3. From the perspective of linear systems, a set  $\{v_1, v_2, \dots, v_n\}$  is called linearly independent set if and only if the system

$A\vec{c} = 0$

has trivial sol, where

$v_1, \dots, v_n$ , and  $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

For example

$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \right\}$

is linearly dependent since there are 4 vectors from  $\mathbb{R}^3$ .

for example:

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} \right\}$

$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Corresponding system is

$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & -1 & -3 & 0 \end{array} \right] \xrightarrow{\substack{-2R_1 + R_2 \\ -2R_1 + R_3}} \left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 5 & 0 \end{array} \right]$

$\xrightarrow{R_2 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

### Linear Independence

Suppose that  $\{v_1, v_2, \dots, v_n\}$  be a set of vectors. Then the given set is said to be a linearly independent set (the vectors are called linearly independent)

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

holds for  $c_1 = c_2 = \dots = c_n = 0$ .

Remark 1: In  $\mathbb{R}^n$ , we can write at least a linearly independent vectors. Any  $n+1$  vectors become linearly dependent.

Remark 2: Suppose that the set  $\{v_1, \dots, v_n\}$  is linearly dependent. Then, there is at least one vector  $v_i$  in this set so that

$$v_i \in \text{span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}.$$

Remark 3: From the perspective of linear systems, a set  $\{v_1, v_2, \dots, v_n\}$  is called linearly independent set if and only if the system

$$A\vec{x} = 0$$

has trivial sol, where

$$A = [v_1 \ v_2 \ \dots \ v_n], \text{ and } \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$



$\{v_1, \dots, v_n\}$   
 there is at least  
 so that  
 $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$   
 of linear systems,  
 called linearly independent  
 system

for example  $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$   
 is linearly dependent since there are  
 4 vectors from  $\mathbb{R}^3$ .

for example:  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Corresponding system is

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & -1 & -3 & 0 \end{array} \right] \begin{array}{l} -2R_1 + R_2 \\ 2R_1 + R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \end{array} \right] \begin{array}{l} R_2 + R_3 \rightarrow R_3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

inf. many solns

||

nonzero sol.

The set is linearly dependent.

and  $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$



for example, what would be  $x$  so that

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & x & 3 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

is linearly independent?

Let  $Q^T + c_1 v_1 + c_2 v_2 = 0$   
 We form the system corresponding to square the matrix

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ -2 & x & 3 & | & -2 \\ 0 & 3 & -5 & | & -1 \end{bmatrix} \xrightarrow{R_1+R_2, R_1} \begin{bmatrix} -3 & x+2 & 4 & | & -2 \\ -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & x-2 & -4 & | & -2 \\ 0 & 3 & -5 & | & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-2 & -4 & | & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-2 & -4 & | & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-2 & -4 & | & -2 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-2-3 & -4+15 & | & -2+3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-5 & 11 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-5 & 11 & | & 1 \end{bmatrix} \xrightarrow{6R_2+R_3} \begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x-5+6 & 11-5 & | & 1-1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & x & 6 & | & 0 \end{bmatrix}$$

Case I  $\alpha=2$

Case II  $\alpha \neq 2$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{-1R_2+R_1} \begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 3 & -5 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

unique sol ✓



Linear Independence

Separate the first  $n-1$  of the  $n$  vectors. Then the quotient is a scalar  $\lambda$  such that  $\lambda \cdot$  (the vector) equals the  $n$ th vector. (The vectors are linearly independent)

consider  $-t \cdot v_1 = 0$   
 hold for  $sc_1 = -c_2 = 0$

Example 1:  $v_1, v_2$  are not at all linearly independent. For the vectors  $v_1, v_2$  they are not.

Example 2: Separate the  $n-1$  vectors. Then, there is at least 1 linearly independent. Then, there is at least one vector  $w$  in this set so that  $v_i \in \text{span}\{v_1, \dots, v_{i-1}, w\}$ .

Example 3: From the perspective of linear systems, a set  $\{v_1, v_2, \dots, v_n\}$  is called linearly independent if and only if the system  $A \cdot \vec{c} = 0$  has only the trivial solution  $\vec{c} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ .

For example

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$$

is linearly dependent since there are 4 vectors from  $\mathbb{R}^3$ .

For example

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Corresponding system is

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & -3 & 3 \end{bmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 + R_1}} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -5 & 2 & 1 \\ 0 & 2 & -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -5 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & -4 & 3 \\ 0 & -5 & 2 & 1 \end{bmatrix}$$

Example, what should be  $\alpha$  so that the given set is linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \alpha \\ 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\} \quad x \in \mathbb{R}$$

Check  $cx_1 + cy_2 = 0$  to find the system consistency to separate  $x$  from  $y$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ -2 & \alpha & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{\substack{R_1+R_2 \\ R_1+R_3 \\ R_1+R_4}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha-2 & -4 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_3} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+R_2} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Case I:  $\alpha=2$   
Case II:  $\alpha \neq 2$

$$\xrightarrow{-LR_2+R_4 \rightarrow R_4} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

unique sol ✓

Case I:  $\alpha=2$   
Unique sol ✓

Case II:  $\alpha \neq 2$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



for example, what should be  $x$  so that  
 the given set  

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$
 is linearly independent?  
 $x \in \mathbb{R}$

Check if  $Q_1 + \alpha Q_2 + Q_3 = 0$   
 by finding the system consistency to represent it  
 after

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & x & -2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 1 & 0 \end{array} \right] \xrightarrow{R_1+R_2, R_1+R_3, R_1+R_4}$$

Case I  $\alpha = 2$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_2}$$

unique sol $\checkmark$

Case II  $\alpha = 2$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

unique sol $\checkmark$

Ex: Let's write some equal statements to  
 "Columns of the matrix  $A$  are linearly independent"

- ① We observe that  $m \geq n$ .
- ②  $Ax=0$  has trivial solution.
- ③  $Ax=b$  has unique sol.



For example, what should be  $\alpha$  so that

the given set

$$\left\{ \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ \alpha \\ 3 \\ -5 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{v_3} \right\}$$

is linearly independent?  $\alpha \in \mathbb{R}$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$   
 We form the system corresponding to augmented matrix

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -2 & \alpha & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & -5 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-2+R_1 \rightarrow R_2 \\ R_1+R_4 \rightarrow R_4}}$$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -4 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow -1/R_1} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array} \right]$$

$$\xrightarrow{-3R_2+R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array} \right]$$

$$\xrightarrow{4R_3+R_4 \rightarrow R_4} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \end{array} \right]$$

Case I:  $\alpha=2$

Case II:  $\alpha \neq 2$

Case I  $\alpha=2$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Unique sol. ✓

Case II  $\alpha \neq 2$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k & 0 & 0 \end{array} \right]$$

$$\xrightarrow{-kR_2+R_4 \rightarrow R_4}$$

Ex. Let's look at statements "Columns"

- ① We observe
- ②  $AX=0$
- ③  $AX=$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow -\frac{1}{k}R_1} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{bmatrix} \quad \text{Case I } \alpha=2$$

unique sol. ✓

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{bmatrix} \quad \text{Case II } \alpha \neq 2$$

→

$$\xrightarrow{-kR_2 + R_4 \rightarrow R_4} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

unique sol ✓

Ex. Lets write some equal statements to "Columns of the matrix A are linearly independent"

- ① We observe that  $m \geq n$ .
- ②  $AX=0$  has trivial solution.
- ③  $AX=b$  has unique sol.

④ In echelon matrix, every column has pivot term.

Columns of A span  $\mathbb{R}^n$ .



for example, what should be  $\alpha$  so that

the given set

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is linearly independent?  $\alpha \in \mathbb{R}$

is  $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$   
 We form the system corresponding to augmented matrix

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & \alpha & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & -5 & -1 & 0 \end{array} \right]$$

$-2R_1 + R_2$   
 $R_1 + R_3$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & \alpha-2 & -4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -4 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\xrightarrow{-3R_2 + R_3} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & \alpha-2 & -4 & 0 \end{array} \right]$$

Case I  $\alpha = 2$

$$\xrightarrow{4R_2 + R_3} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \end{array} \right]$$

Case I.  $\alpha = 2$   
 Case II.  $\alpha \neq 2$

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha-2 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{-\alpha R_2 + R_4} \left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

unique sol ✓

Ex. Let's write some equal statements to "Columns of the matrix  $A$  are linearly independent"

- ① We observe that  $m \geq n$ .
- ②  $Ax=0$  has trivial solution
- ③  $Ax=b$  has unique sol.

④ In echelon matrix, every column has pivot term.

Columns of  $A$  span  $\mathbb{R}^n$ .

## Linear transformations

### Matrix Calculus

#### Inverse of a matrix

Def: ( $L^{-1}$ ) Suppose that  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then,  $T$  is said to be  $L^{-1}$  if and only if  $Tx = Ty$  when  $x = y$

Alternatively, we can define being  $L^{-1}$  in a better form. We need to define null space (kernel) of transformation.

Def: The null space of a transformation is a set given by  
Null  $T$  or  $\text{Ker } T = \{x \in \mathbb{R}^n : Tx = \vec{0}\}$ .

$\triangle$   $T$  is  $L^{-1}$  if and only if  $\text{Ker}(T) = \{\vec{0}\}$ .

Ex 1: Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is  $T$   $L^{-1}$ ? No

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x+y=0 \\ x-y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$$

$$\text{Ker } T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Ex 2:  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be a linear transformation given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is  $T$   $L^{-1}$ ?

$\{\vec{0}\}$

Ex 2:  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is  $T$  1-1?

Let's write the matrix of transformation

$$A_{4 \times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transformation, we need see  $\text{Ker}(T) = \{\vec{0}\}$ . Equivalently, we can check

Does the system  $Ax=0$  have trivial sol

We immediately form the augmented matrix

$$\left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \xrightarrow[-R_1 + R_3]{-2R_1, R_2} \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

$=0$   
 $=0$

$\boxed{z?}$

on  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{\vec{0}\}$

⚠  $T$  is 1-1 if and only if  $\text{Ker}(T) = \{0\}$ .

Ex 1: Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is  $T$  1-1? No

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x+y=0 \\ x-y=0 \end{cases} \begin{cases} x=0 \\ y=0 \end{cases} \quad \boxed{z?}$$

$$\text{Ker } T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{0\}$$

Ex 2:  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is  $T$  1-1?

Let's write the matrix of transformation

$$A_{4 \times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transformation, we need see  $\text{Ker}(T) = \{0\}$ . Equivalently, we can check

Does the system  $Ax=0$  have trivial sol

We immediately form the augmented matrix

$$\left[ \begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow[-R_1 + R_3]{-2R_1 + R_2} \left[ \begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

$x_2=0$  and  $x_1=0$   
Trivial sol for  $Ax=0$

$$\text{Ker}(T) = \{0\}$$

$T$  is 1-1

⚠  $T$  is 1-1 if and only if  $\text{Ker}(T) = \{\vec{0}\}$ .

Ex 1: Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix}$$

Is  $T$  1-1? No

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x+y=0 \\ x-y=0 \end{cases} \begin{cases} x=0 \\ y=0 \end{cases} \quad \boxed{z?}$$

$$\text{Ker } T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{\vec{0}\}$$

Ex 2:  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be a linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x-y \\ x \\ y \end{pmatrix}$$

Is  $T$  1-1?

Let's write the matrix of transformation

$$A_{4 \times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transformation, we need see  $\text{Ker}(T) = \{\vec{0}\}$ . Equivalently, we can check

does the system  $Ax=0$  have trivial sol.

We immediately form the augmented matrix

$$\left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{-2R_1+R_2 \\ -R_1+R_3}} \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{array} \right]$$



mation

T-F: If the columns of the standard matrix of linear transformation  $T$  are linearly independent, then  $T$  is 1-1

Ex 1: Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Is  $T$  1-1? No

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x+y=0 \\ x-y=0 \end{cases} \begin{matrix} x=0 \\ y=0 \end{matrix} \quad \boxed{z?}$$

$$\text{Ker } T = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \{ \vec{0} \}$$

write  
 $\frac{6}{7}R_2 + R_3 \rightarrow R_3$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right]$$

Ex 2:  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$  transformation given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} =$$

Is  $T$  1-1?

Let's write the matrix

$$A_{4 \times 2} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To have a 1-1 transform  
 $\text{Ker}(T) = \{ \vec{0} \}$  Equiv

Ex 3:  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation

given by  $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y + 3z \\ 3x - y + 2z \\ -x + 4y - 3z \end{pmatrix}$

Is  $T$  1-1? Yes

The matrix of transformation

$$A_{3 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ -1 & 4 & -3 \end{bmatrix}$$

Then, we'll solve  $Ax=0$ , and therefore we write

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & -1 & 2 & 0 \\ -1 & 4 & -3 & 0 \end{array} \right] \xrightarrow{\substack{-3R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3}} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 6 & 0 & 0 \end{array} \right] \xrightarrow{\substack{6/7 R_2 + R_3 \rightarrow R_3 \\ 7R_2 + R_3 \rightarrow R_3}}$$

unique sol.  
zero sol.

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right]$$

T-F: If the column transformation  $T$  or

Ex 1: Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y + 3z \\ 3x - y + 2z \\ -x + 4y - 3z \end{pmatrix}$$

Is  $T$  1-1? Yes

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y + 3z \\ 3x - y + 2z \\ -x + 4y - 3z \end{pmatrix}$$

$\text{Ker } T = \{0\}$

Defn: A linear transformation  $T$  is said to be onto if  $\text{codomain}(T) = \text{Range}(T)$ .

Equivalently,  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. If the columns of the matrix  $A$  for  $T$  span  $\mathbb{R}^m$ , then  $T$  is onto!

for example:  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \\ -x-y-2z \end{pmatrix}$$

Is  $T$  onto?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

That is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

That is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ v \\ w \end{bmatrix}$$

We form the augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & v \\ -1 & -1 & -2 & w \end{array} \right] \text{ We want that this system is consistent!}$$

$$\begin{array}{l} R_1 + R_3 \rightarrow R_3 \\ \rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & v \\ 0 & -2 & -2 & u+w \end{array} \right] \xrightarrow{2R_2 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & v \\ 0 & 0 & 0 & u+2v+w \end{array} \right] \end{array}$$

inconsistent

Since the columns could not span  $\mathbb{R}^3$  and since we have 3 columns, this means columns are dependent. So  $T$  is also not 1-1.

for example  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+2z \end{pmatrix}$$

Is  $T$  1-1 and onto?

sol. The matrix of the transformation

is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

? Can  $Ax=0$  have unique sol.?

Defn: A linear transformation  $T$  is said to be onto if  $\text{codomain}(T) = \text{Range}(T)$ .

Equivalently,  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. If the columns of the matrix  $A$  for  $T$  span  $\mathbb{R}^m$ , then  $T$  is onto!

for example:  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ y + z \\ -x - y - 2z \end{pmatrix}$$

Is  $T$  onto?

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & -2 \end{bmatrix}$$

That is  
 $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Defn: A linear transformation  $T$  is said to be onto if  $\text{codomain}(T) = \text{Range}(T)$ .

Equivalently,  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. If the columns of the matrix  $A$  for  $T$  span  $\mathbb{R}^m$ , then  $T$  is onto!

for example:  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y+z \\ -x-y-2z \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

That is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

We form the augmented matrix

$$\left[ \begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ -1 & -1 & -2 & w \end{array} \right]$$

We want that this system is consistent

$$\xrightarrow{R_1+R_3} \left[ \begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ 0 & -2 & -2 & u+w \end{array} \right] \xrightarrow{2R_2+R_3} \left[ \begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & 1 & v \\ 0 & 0 & 0 & u+3v \end{array} \right]$$

inconsistent

Since the columns could not span  $\mathbb{R}^3$  and since we have 3 columns, this means columns are dependent. So  $T$  is also not 1-1

for example  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+2z \end{pmatrix}$$

Is  $T$  1-1 and onto?

sol: The matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Can  $Ax$  have unique sol?

Not 1-1

No!

To check onto property we pick arbitrary vector  $\begin{bmatrix} u \\ v \end{bmatrix}$  and write  $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ v \\ w \end{bmatrix}$$

augmented matrix

$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$  We want that this system is consistent!

$$\left[ \begin{array}{ccc|c} 0 & 1 & -2 & u \\ 1 & 0 & 0 & v \\ -2 & 0 & 0 & w \end{array} \right] \xrightarrow{2R_2 + R_3 \rightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -1 & 0 & u \\ 0 & 1 & -2 & v \\ 0 & 0 & 0 & u+2v+w \end{array} \right]$$

inconsistent

columns could not span  $\mathbb{R}^3$  and since we have means columns are dependent. So  $T$  is also not 1-1.

for example:  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+2z \end{pmatrix}$$

Is  $T$  1-1 and onto?

sol. The matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

$\triangle ?$  Can  $Ax=0$  have unique sol?

T-F: If  $T$  is 1-1, then  $T$  is onto (F)

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \\ 2x \end{pmatrix}$

Classwork: Is  $T$  1-1 and onto?

$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$  not onto!

$T$  is 1-1

To check 1-1,  $Ax=0$

$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{array} \right] \xrightarrow{R_1+R_2-R_3} \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array} \right]$   
 $x_1=0, x_2=0$

Concluding remarks

- If  $T$  is 1-1 and onto, then it is called a bijection.
- If  $T$  is a bijection, then it has inverse.

Matrix Calculus

We can express an  $m \times n$  matrix  $A$  in the form  $[a_{ij}]$ .

$i=1, \dots, m, j=1, \dots, n$

Transpose: Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. The transpose of  $A$ , denoted by  $A^T$ , is an  $n \times m$  matrix given by  $A^T = [a_{ji}]$ .

Precisely, we make rows columns of the new matrix while we make columns rows.

row  $\rightarrow$  column

$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \\ -4 & 7 \\ 2 & 0 \end{bmatrix}$

Transpose  
Substitution  
Pattern  
Multiplication  
Dense



Scalar multiplication: For any matrix  $A_{m \times n} = [a_{ij}]$

We can define scalar multiply

$cA_{m \times n} = [ca_{ij}]$  for  $c \in \mathbb{R}$ .  
(multiply all entries with  $c$ )

Like

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

then

$$4A = \begin{bmatrix} 8 & 4 & 0 \\ -4 & 12 & 8 \end{bmatrix}$$



$$(cA)^T = cA^T \text{ for } c \in \mathbb{R}$$

Matrix addition Suppose that  $A = [a_{ij}]$

and  $B = [b_{ij}]$  be two  $m \times n$  matrices.

Then

$$A + B = C = [c_{ij}] = [a_{ij} + b_{ij}]$$

Addition is term by term.

△ Precisely, we mean rows (columns) of the two matrix like were taking separate

Transpose  
Subtraction  
Addition  
Dense

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{rows}} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix} \xrightarrow{\text{rows}} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rows}} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

T-F: If  $T$  is 1-1, then  $T$  is onto (F)

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y \\ 2x \end{pmatrix}$

Classwork: Is  $T$  1-1 and onto?

$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$  not onto!

To check 1-1,  $Ax=0$

$\left[ \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{array} \right] \xrightarrow{\substack{R_1+R_2 \\ R_2+R_3}} \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array} \right] \xrightarrow{R_3-2R_2} \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$   
 $x_2=0$   
 $x_1=0$

$T$  is 1-1

Concluding remarks:

- \* If  $T$  is 1-1 and onto, then it is called a bijection
- \* If  $T$  is a bijection, then it has inverse

Matrix Calculus

We can express an  $m \times n$  matrix  $A$  in the form  $[a_{ij}]$ .

$i=1, \dots, m, j=1, \dots, n$

Transpose: Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. The transpose of  $A$ , denoted by  $A^T$ , is an  $n \times m$  matrix given by  $A^T = [a_{ji}]$ .

Precisely, we make rows columns of the new matrix while we are taking transpose.

the determinant is not

$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 \\ -1 & 7 \\ 2 & 0 \end{bmatrix}$

Transpose  
Scalar  
Addition  
Matrix  
Multiplication  
Inverse

$$\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ v \\ w \end{bmatrix}$$

augmented matrix

$$\begin{bmatrix} 4 \\ v \\ w \end{bmatrix} \text{ We want that this system is consistent!}$$

$$\begin{bmatrix} 0 & 4 \\ 1 & v \\ -2 & w \end{bmatrix} \xrightarrow{2R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & | & u \\ 0 & 1 & 1 & | & v \\ 0 & 0 & 0 & | & u+2v+w \end{bmatrix}$$

inconsistent

columns could not span  $\mathbb{R}^3$  and since we have means columns are dependent. So  $T$  is also not 1-1.

for example  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+2z \end{pmatrix}$$

Is  $T$  1-1 and onto?

Sol: The matrix of the transformation is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

Can  $Ax=0$  have unique sol? No!

Not 1-1!

To check onto property, let's pick an arbitrary vector

$\begin{bmatrix} 4 \\ v \end{bmatrix}$  and write

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ v \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & | & 4 \\ -1 & 0 & 2 & | & v \end{bmatrix} \xrightarrow{\text{R}_2 + \text{R}_1} \begin{bmatrix} 1 & -1 & 1 & | & 4 \\ 0 & -1 & 3 & | & u+v \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & | & u \\ 0 & -1 & 3 & | & u+v \end{bmatrix}$$

if many solns!  
consistent!

$T$  is onto!

Matrix addition Suppose that  $A = [a_{ij}]$   
and  $B = [b_{ij}]$  be two  $m \times n$  matrices.

Then

$$A + B = C = [c_{ij}] = [a_{ij} + b_{ij}]$$

Addition is term by term.

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ 2 & 1 & -3 \\ -3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_A \quad \underbrace{\hspace{10em}}_B \quad \underbrace{\hspace{10em}}_C$

⚠  $(A+B)^T = A^T + B^T$

Properties

$$c(A+B) = cA + cB$$

$$A+B = B+A$$

$$(A+B)+C = A+(B+C)$$

Matrix Multiplication

Important: There is a very restrictive condition in matrix multiplication according to sizes.

$$A_{m \times n} B_{n \times p} = C_{m \times p}$$

i.e

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

⚠  $AB \neq BA$

Transpose ✓  
Scalar multiply ✓  
Addition ✓  
Multiplication ✓  
Inverse ✓

$AB$  X

$BA$  ✓

$BC$  X

$CB$  ✓

$BD$  ✓

$DB$  X

$AC$  ✓

$CA$  ✓

$CB$  ✓

### Scalar multiplication: For any matrix $A_{m \times n} = [a_{ij}]$

We can define scalar multiply

$$cA_{m \times n} = [ca_{ij}] \text{ for } c \in \mathbb{R}$$

(multiply all entries with  $c$ )

Like

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

then

$$4A = \begin{bmatrix} 8 & 4 & 0 \\ -4 & 12 & 8 \end{bmatrix}$$

$$(cA)^T = cA^T \text{ for } c \in \mathbb{R}$$



### Matrix addition: Suppose that $A = [a_{ij}]$

and  $B = [b_{ij}]$  be two  $m \times n$  matrices:

Then

$$A + B = C = [c_{ij}] = [a_{ij} + b_{ij}]$$

Addition is term by term.

Precisely, we make rows columns of the new matrix while being careful

Transpose  
Addition  
Multiplication  
Distributive

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 7 \\ -4 & 1 & 4 \\ 2 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & -4 & 2 \\ 2 & 1 & 0 \\ 7 & 4 & 0 \end{bmatrix}$$

## Computing $AB$

$$A_{m \times n} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \text{ where } R_i \text{ is a row of } A.$$

$$B_{n \times p} = [C_1 \ C_2 \ \dots \ C_p], \text{ where } C_i \text{ is a column of } B.$$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 & \dots & R_1 C_p \\ R_2 C_1 & R_2 C_2 & \dots & \dots & R_2 C_p \\ \vdots & \vdots & \vdots & \dots & \vdots \\ R_m C_1 & R_m C_2 & \dots & \dots & R_m C_p \end{bmatrix}_{m \times p}$$

Ex:

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} \quad \quad \quad \\ \quad \quad \quad \\ \quad \quad \quad \end{bmatrix}_{3 \times 2}$$

Matrix addition Suppose that  $A=[a_{ij}]$   
and  $B=[b_{ij}]$  be two  $m \times n$  matrices.

Then

$$A+B=C = [c_{ij}] = [a_{ij}+b_{ij}]$$

$m \times n$     $m \times n$     $m \times n$

Addition is term by term.

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ 2 & 1 & -3 \\ -3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$3 \times 3$     $3 \times 3$     $3 \times 3$

A                  B                  C

⚠  $(A+B)^T = A^T + B^T$

Properties

$$c(A+B) = cA + cB$$

$$A+B = B+A$$

$$(A+B)+C = A+(B+C)$$

Matrix Multiplication

Important There is a very  
restrictive condition in matrix  
multiplication according to sizes.

$$A_{m \times n} B_{n \times p} = C_{m \times p}$$

Ex

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

⚠  $AB \neq BA$

Transpose ✓  
Scalar mult. ✓  
Addition ✓  
Multiplication ✓  
Inverse

$AB$  X  
 $3 \times 2$   $3 \times 3$

$BA$  ✓  
 $3 \times 3$   $3 \times 2$

$BC$  X  
 $3 \times 3$   $2 \times 3$

$CB$  ✓  
 $2 \times 3$   $3 \times 3$

$BD$  ✓  
 $3 \times 3$   $3 \times 4$

$DB$  X  
 $3 \times 3$   $3 \times 2$

$AC$  ✓  
 $3 \times 2$   $2 \times 3$

$CA$  ✓  
 $2 \times 3$   $3 \times 2$

# Computing AB

$$A_{m \times n} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \text{ where } R_i \text{ is a row of } A.$$

$$B_{n \times p} = [C_1 \ C_2 \ \dots \ C_p], \text{ where } C_i \text{ is a column of } B.$$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & \dots & R_1 C_p \\ R_2 C_1 & R_2 C_2 & \dots & R_2 C_p \\ \vdots & \vdots & \ddots & \vdots \\ R_m C_1 & R_m C_2 & \dots & R_m C_p \end{bmatrix}_{m \times p}$$

Ex 1

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 3 & 1 \\ -4 & 9 \\ -3 & 17 \end{bmatrix}_{3 \times 2}$$

$$CA = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 8 & -9 \\ -1 & 3 \end{bmatrix}_{2 \times 2}$$

$$BD = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}_{3 \times 4} = \begin{bmatrix} 4 & 9 & 5 & -5 \\ 2 & 4 & 6 & 7 \\ 12 & 6 & 13 & 11 \end{bmatrix}_{3 \times 4}$$

Properties  
 $C(A+B) = CA + CB$   
 $A+B = B+A$

$$(A+B) + C = A + (B+C)$$

Matrix Multiplication

Important: There is a very restrictive condition in matrix multiplication.

$$A_{m \times n} B_{n \times p} = C_{m \times p}$$

Properties  
 Additive  
 Associative  
 Multiplicative  
 Inverse

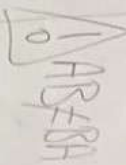
$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- AB ✓
- BA ✓
- CA ✓
- CB ✓
- CD ✓
- DC ✓
- AD ✓
- DA ✓
- AC ✓
- CA ✓
- AD ✓
- DA ✓
- AC ✓
- CA ✓





- Inverse of a matrix
- LU-factorization } Slides
- Determinants

Identity matrix =  $I_{n \times n} = \begin{cases} a_{ij} = 1 & \text{if } i=j \\ a_{ij} = 0 & \text{if } i \neq j \end{cases}$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$A^{-1}$  = inverse of  $A$

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = ad - bc$$

Property 5

$$A^{-1} A x = A^{-1} b$$

$$I_{n \times n} x = A^{-1} b$$

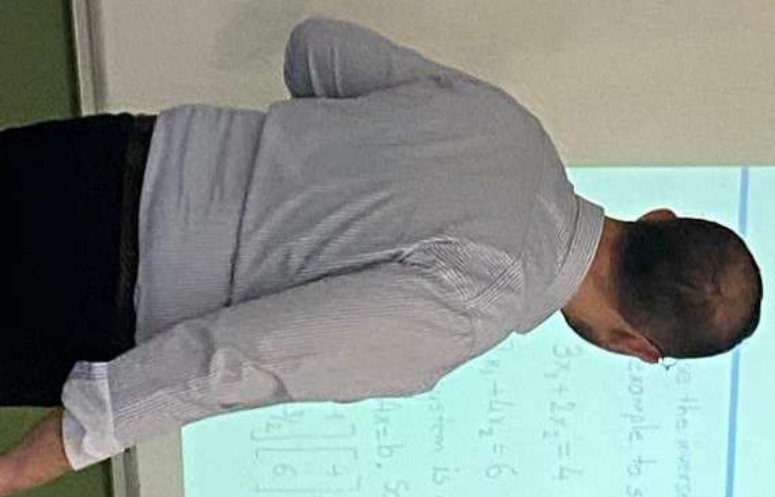
$$x = A^{-1} b$$

invertible  $\equiv$  nonsingular  
noninvertible  $\equiv$  singular

If  $B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ , then

$$\det B = 2 \cdot 1 - (-1) \cdot 3 = 5$$

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix}$$



the inverse of the matrix  $A$   
example to solve the system

$$\begin{cases} 3x + 2y = 4 \\ x + 4y = 6 \end{cases}$$

system is equivalent to

$$A^{-1} b = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

✓

### Computing AB

$$A_{m \times n} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}, \text{ where } R_i \text{ is a row of } A.$$

$$B_{n \times p} = [C_1 \ C_2 \ \dots \ C_p], \text{ where } C_i \text{ is a column of } B.$$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 & \dots & R_1 C_p \\ R_2 C_1 & R_2 C_2 & \dots & \dots & R_2 C_p \\ \vdots & \vdots & \vdots & \dots & \vdots \\ R_m C_1 & R_m C_2 & \dots & \dots & R_m C_p \end{bmatrix}_{m \times p}$$

Ex:

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 3 & 1 \\ -4 & 9 \\ -3 & 17 \end{bmatrix}_{3 \times 2}$$

$$CA = \begin{bmatrix} 2 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 8 & -9 \\ -1 & 3 \end{bmatrix}_{2 \times 2}$$

$$BD = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 2 & 1 & 7 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 4 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}_{3 \times 4} = \begin{bmatrix} 2 & 1 & 3 & -2 \\ 2 & 5 & 8 & 1 \\ 2 & 1 & 7 & 1 \end{bmatrix}_{3 \times 4}$$

Properties

$$c(A+B) = cA + cB$$

$$A+B = B+A$$

$$(A+B)+C = A+(B+C)$$

Matrix Multiplication

Important: There is a very restrictive condition in matrix multiplication: multiply according to sizes

$$A_{m \times n} B_{n \times p} = C_{m \times p}$$

$$I_c$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

it exists!

We need to write

[A is invertible iff A is row equivalent to I<sub>n</sub>]

Find A<sup>-1</sup> if

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_2+R_1 \\ R_3-2R_1}} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & -2 & 0 & 1 \end{array} \right]$$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = b$$

$$x = A^{-1}b$$

$$(AB)^T = B^T A^T$$

$$(A^{-1})^{-1} = A$$

### ELEMENTARY MATRICES

- Property 7: An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .
- ALGORITHM FOR FINDING  $A^{-1}$ :
  - Row reduce the augmented matrix  $[A \mid I]$ . If  $A$  is row equivalent to  $I$ , then  $[A \mid I]$  is row equivalent to  $[I \mid A^{-1}]$ . Otherwise,  $A$  does not have an inverse.

4: Use the inverse of the matrix A  
from the previous example to solve the system

$$3x_1 + 2x_2 = 4$$

$$7x_1 + 4x_2 = 6$$

This system is equivalent to

$$Ax = b, \text{ so,}$$

$$\begin{bmatrix} 2 & 1 \\ 3/2 & -3/2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -8+6 \\ 14-9 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \checkmark$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

It exists!

We need to write

[A is invertible iff A is row equivalent to I<sub>n</sub>]

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det A \rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 5 & -1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_1+R_2 \rightarrow R_2 \\ -2R_1+R_3 \rightarrow R_3}} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 5 & -1 & 0 & -2 & 1 \end{array} \right]$$

$$\xrightarrow{-5R_2+R_3 \rightarrow R_3} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -6 & -7 & -5 & 1 \end{array} \right] \xrightarrow{-1/6 R_3} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array} \right]$$

$$\xrightarrow{-R_3+R_2 \rightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array} \right] \xrightarrow{R_2+R_1 \rightarrow R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 5/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array} \right]$$

### ELEMENTARY MATRICES

Property 7: An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .

#### ALGORITHM FOR FINDING $A^{-1}$ :

Row reduce the augmented matrix  $[A \ I]$ . If  $A$  is row equivalent to  $I$ , then  $[A \ I]$  is row equivalent to  $[I \ A^{-1}]$ . Otherwise,  $A$  does not have an inverse.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

It exists!

We need to write

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_1+R_2-R_3 \\ -2R_1+R_3-R_1}} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 5 & -1 & 0 & -2 & 0 \end{array} \right]$$

$$\xrightarrow{-5R_2+R_3-R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -6 & -2 & -5 & 1 \end{array} \right] \xrightarrow{-1/6 R_3} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 5/6 & -1/6 \end{array} \right]$$

$$\xrightarrow{-R_2+R_3-R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array} \right] \xrightarrow{R_2+R_1-R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 5/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 7/6 & 5/6 & -1/6 \end{array} \right]$$

Find  $A^{-1}$  if

[A is invertible iff A's row equivalent to  $I_n$ ]

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 7 & 5 & -1 \end{bmatrix}$$

Suppose that  $A_n$  is invertible. Then

- i) A is row equivalent to  $I_n$
- ii)  $Ax=b$  has unique sol.
- iii)  $Ax=0$  has trivial sol.
- iv) A has n pivot elems
- v) Columns of A span  $\mathbb{R}^n$
- vi) Columns of A are linearly independent
- vii) The linear transformation  $Tx=Ax$  is 1-1 and onto (by (iii) and (iv))
- viii) The linear transformation  $Tx=Ax$  has inverse, where  $T^{-1}x=A^{-1}x$ .

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

it exists!

We need to write

[A is invertible iff A is row equivalent to I<sub>n</sub>]

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = ad-bc$$

Property 5

$$A^{-1} A x = A^{-1} b$$

$$I_n x = A^{-1} b$$

$$x = A^{-1} b$$

Remember:  $(AB)^T = B^T A^T$

Similarly:  $(AB)^{-1} = B^{-1} A^{-1}$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{R_1+R_2 \\ -2R_1+R_3}} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 5 & -1 & -2 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{-5R_2+R_3} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & -7 & -5 & 1 \end{array} \right]$$

### ELEMENTARY MATRICES

- Property 7: An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A$  to  $I_n$  also transforms  $I_n$  into  $A^{-1}$ .
- ALGORITHM FOR FINDING  $A^{-1}$ :
  - Row reduce the augmented matrix  $[A \mid I]$  if  $A$  is row equivalent to  $I$ , then  $[A \mid I]$  is row equivalent to  $[I \mid A^{-1}]$ . Otherwise,  $A$  does not have an inverse.

$$= \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 1 \\ -1 & 1 & 1 \\ 7 & 5 & -1 \end{bmatrix}$$

Suppose that  $A_{n \times n}$  is invertible.  
Then

- i)  $A$  is row equivalent to  $I_n$
- ii)  $Ax=b$  has unique sol.
- iii)  $Ax=0$  has trivial sol.
- iv)  $A$  has  $n$  pivot elems
- v) Columns of  $A$  span  $\mathbb{R}^n$
- vi) Columns of  $A$  are linearly independent.
- vii) The linear transformation  $Tx=Ax$  is 1-1 and onto (by (iii) and (v))
- viii) The linear transformation  $Tx=Ax$  has inverse, where  $T^{-1}x=A^{-1}x$ .



Is the matrix A invertible  
if

$$A = \begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 3 & -1 & -1 \end{bmatrix} \xrightarrow{\substack{2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3}} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\xrightarrow{-5/8 R_2 \rightarrow R_2} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\xrightarrow{5R_2 \rightarrow R_3} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A is singular.

inverse matrix ✓  
factorization  
determinant

factorization:  $A = BC$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \xrightarrow{\text{col 1}} \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \xrightarrow{\text{col 2}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \xrightarrow{\text{col 3}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

### AN LU FACTORIZATION ALGORITHM

- Compute the first columns of A and L. The row operations that create zeros in the first column of A will also create zeros in the first column of L.
- To make this same correspondence of row operations on A hold for the rest of L, which is row reduction of A to an echelon form U. That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform A into U.

$$A = \begin{bmatrix} 2 & 1 & -1 & 4 & -2 \\ -4 & -3 & -4 & 1 & -1 \\ 1 & -5 & -4 & 1 & 8 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\substack{2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3}} \begin{bmatrix} 2 & 1 & -1 & 4 & -2 \\ 0 & -5 & -6 & 9 & -1 \\ 3 & -4 & -5 & 5 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = A_1$$

$$A_1 \xrightarrow{-5/8 R_2 \rightarrow R_2} \begin{bmatrix} 2 & 1 & -1 & 4 & -2 \\ 0 & 1 & -3/4 & 9/8 & -1/8 \\ 3 & -4 & -5 & 5 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = A_2$$

$$A_2 \xrightarrow{5R_2 \rightarrow R_3} \begin{bmatrix} 2 & 1 & -1 & 4 & -2 \\ 0 & 1 & -3/4 & 9/8 & -1/8 \\ 0 & -1 & -5 & 5 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = A_3$$

$$A_3 \xrightarrow{R_1+R_2 \rightarrow R_1} \begin{bmatrix} 2 & 2 & -7/4 & 35/8 & -5/8 \\ 0 & 1 & -3/4 & 9/8 & -1/8 \\ 0 & -1 & -5 & 5 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = A_4$$

$$A_4 \xrightarrow{R_1 - R_2 \rightarrow R_1} \begin{bmatrix} 2 & 1 & -1 & 4 & -2 \\ 0 & 1 & -3/4 & 9/8 & -1/8 \\ 0 & -1 & -5 & 5 & 6 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = U$$

Is the matrix A invertible

if

$$A = \begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -4 \\ 2 & 4 & 0 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{\substack{2R_1+R_2 \\ R_1+R_3 \rightarrow R_3}} \begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\xrightarrow{-5/8R_2+R_3 \rightarrow R_3}$$

$$\begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

A is singular

inverse matrix ✓  
factorization  
determinant

### MATRIX FACTORIZATION

- A factorization expresses A as a product of more matrices.
- Whereas matrix factorization involves a synthesis of transformations.



Write the LU factorization  
for the following matrices

$$A = \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ -6 & -3 & 5 & 1 & 2 \\ -3 & 2 & 0 & 1 & -1 \end{bmatrix}$$

$$A \xrightarrow[\substack{2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3}]{\substack{3 \quad -1 \quad 4 \quad 2 \quad 0 \\ 0 \quad -5 \quad 13 \quad 5 \quad 2 \\ 0 \quad 0 \quad 3/5 \quad 4 \quad -3/5}} \xrightarrow{1/5 R_2 \rightarrow R_2} \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -1 & 13/5 & 5/5 & 2/5 \\ 0 & 0 & 3/5 & 4 & -3/5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \rightarrow \text{col 1 of } L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1/5 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 3 \\ -6 & -1 & 4 \end{bmatrix} \xrightarrow[\substack{1/2 R_1 + R_2 \rightarrow R_2 \\ -3/2 R_1 + R_3 \rightarrow R_3}]{\substack{-4 \quad 2 \quad 2 \\ 0 \quad 2 \quad 4 \\ 0 \quad -4 \quad 1}} \xrightarrow{1/2 R_2 \rightarrow R_2} \begin{bmatrix} -4 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 3/2 \end{bmatrix} \rightarrow \text{col 1 of } L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 3/2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightarrow \text{col 2 of } L$$

$$B_3 \rightarrow \begin{bmatrix} -1 & 2 & -4 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverse matrix ✓

Factorization:  $A=BC$

$$A = \begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \frac{1}{2}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ -3 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \leftarrow \text{col 1}$$

$$\begin{bmatrix} 1 \\ -9 \\ 12 \\ 3 \end{bmatrix} \frac{1}{3} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 1 \end{bmatrix} \leftarrow \text{col 2}$$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \\ 2 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 1 \\ 2 \\ 0.5 \\ 1 \end{bmatrix} \leftarrow \text{col 3}$$

### AN LU FACTORIZATION ALGORITHM

- Compare the first columns of  $A$  and  $L$ . The row operations that create zeros in the first column of  $A$  will also create zeros in the first column of  $L$ .
- To make this same correspondence of row operations on  $A$  hold for the rest of  $L$ , watch a row reduction of  $A$  to an echelon form  $U$ . That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform  $A$  into  $U$ .

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 7 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & -9 & 8 & -13 & 10 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_1$$

$$\xrightarrow{\sim} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & -9 & 8 & -13 & 10 \\ 0 & 0 & -12 & 12 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Write the LU factorization for the following matrices

$$A = \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ -6 & -3 & 5 & 1 & 2 \\ -3 & 2 & 0 & 1 & -1 \end{bmatrix}$$

$$A \xrightarrow[\substack{2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3}]{\substack{3 \quad -1 \quad 4 \quad 2 \quad 0 \\ 0 \quad -5 \quad 13 \quad 5 \quad 2 \\ 0 \quad 1 \quad 4 \quad 3 \quad -1}} \xrightarrow{\substack{1/5 R_2 \rightarrow R_2 \\ 1/5 R_3 \rightarrow R_3}} \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -1 & 13/5 & 1 & 2/5 \\ 0 & 1 & 4 & 3 & -1 \end{bmatrix}$$

U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1/5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \rightarrow \text{col of } L$$

$$B = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 3 \\ -6 & -1 & 4 \end{bmatrix} \xrightarrow[\substack{1/2 R_1 + R_2 \rightarrow R_2 \\ -3/2 R_1 + R_3 \rightarrow R_3}]{\substack{-4 \quad 2 \quad 2 \\ 0 \quad 2 \quad 4 \\ 0 \quad -4 \quad 1}} \xrightarrow[\substack{2R_2 + R_3 \rightarrow R_3}]{\substack{-4 \quad 2 \quad 2 \\ 0 \quad 2 \quad 4 \\ 0 \quad 0 \quad 9}}$$

$$\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 3/2 \end{bmatrix} \rightarrow \text{1st col of } L$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightarrow \text{2nd col of } L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 3/2 & -2 & 1 \end{bmatrix}$$

$$+R_2 \rightarrow R_2 \quad \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 4 & 4 & 3 & -1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 0 & \frac{37}{5} & 4 & -\frac{3}{5} \end{bmatrix}$$

Find an LU-factorization for  $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1/5 & 1 \end{bmatrix}$

Solution: The reduction to row-echelon form is  $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{2R_2 + R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = U$

$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$

Hence  $A = LU$  where  $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

$L_2$  of  $L$



Write the LU Factorization for the following matrices

$$A = \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ -6 & -3 & 5 & 1 & 2 \\ -3 & 2 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 2 & 2 \\ 2 & 1 & 3 \\ -6 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \xrightarrow{\text{col 1}} \begin{bmatrix} -4 \\ -1/2 \\ 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{\text{col 2}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$$

$$A \xrightarrow{2R_1+R_2} \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 1 & 4 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} \xrightarrow{\text{col 1 of L}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \\ 0 & -4 \end{bmatrix} \xrightarrow{1/2 R_1+R_2} \begin{bmatrix} 4 & 2 \\ 0 & -4 \\ 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 9 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} -4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 9 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -1 & 4 & 2 & 0 \\ 0 & -5 & 13 & 5 & 2 \\ 0 & 0 & 13/5 & 4 & -3/5 \end{bmatrix}$$

Find an LU-factorization for A =

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + 4x_2 + 4x_3 &= 4 \\ x_2 - 2x_3 &= -2 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

$$\left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-R_1+R_3 \\ -R_2+R_3}}$$

$$\left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array} \right] \xrightarrow{R_3+R_2}$$

$$\left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array} \right] \xrightarrow{\text{unique sol.}}$$

$$\left[ \begin{array}{ccc|c} 0 & -x & y & 6 \\ 2 & -6 & 2 & -2 \\ 0 & -3 & x^2 & 2 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & x^2 & y \\ 2 & -6 & 2 & -2 \\ 0 & -x & y & 6 \end{array} \right] \xrightarrow{-2R_1+R_2}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & x^2 & y \\ 0 & 0 & 2-2x^2 & -2-2y \\ 0 & -x & y & 6 \end{array} \right] \xrightarrow{\substack{1/2 R_2 \\ R_2 \leftrightarrow R_3}}$$

$$\left[ \begin{array}{ccc|c} 1 & -3 & x^2 & y \\ 0 & -x & y & 6 \\ 0 & 0 & 1-x^2 & -1-y \end{array} \right]$$

$$(1-x^2)z = -(1+y)$$

No sol. case

$$1-x^2=0 \text{ and } -1-y \neq 0$$

If  $x=1$ , then no sol.

Inf. many sol. case

$$1-x^2=0 \text{ and } -1-y=0$$

If  $x=1$ , then inf. many sol.

Determine the values of  $a$  and  $b$  so that the following system

$$\begin{aligned} -9x + 4z &= 1 \\ 2x - 5y + 2z &= -3 \\ x - 2y + 3z &= 7 \end{aligned}$$

- i. has infinitely many solutions.
- ii. has no solution.
- iii. has a unique solution.



$$R_1 \quad 2x_1 + 4x_2 + 4x_3 = 4$$

$$x_2 - 2x_3 = -2$$

$$2x_1 + 3x_2 = 0$$

$$\left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array} \right] \xrightarrow{-R_1+R_3, -R_3} \left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array} \right]$$

$$\xrightarrow{R_2+R_3, R_3} \left[ \begin{array}{ccc|c} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array} \right]$$

$$\xrightarrow{R_1 \div 2} \left[ \begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array} \right] \text{ unique sol.}$$

$$\left[ \begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{ccc|c} 0 & -\alpha & 4 & 6 \\ 2 & -6 & 2 & -2 \\ 0 & -\alpha & 4 & 6 \end{array} \right]$$

$$\xrightarrow{-2R_1+R_2} \left[ \begin{array}{ccc|c} 1 & -3 & y^2 & y \\ 0 & -\alpha & 4 & 6 \\ 0 & -\alpha & 4 & 6 \end{array} \right] \xrightarrow{-R_2+R_3} \left[ \begin{array}{ccc|c} 1 & -3 & y^2 & y \\ 0 & 0 & 2-2y^2 & -2-2y \\ 0 & -\alpha & 4 & 6 \end{array} \right]$$

$$\xrightarrow{1/2 R_2 \leftrightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -3 & y^2 & y \\ 0 & -\alpha & 4 & 6 \\ 0 & 0 & 1-y^2 & -1-y \end{array} \right]$$

$$(1-y^2)z = -(1+y)$$

No sol. case

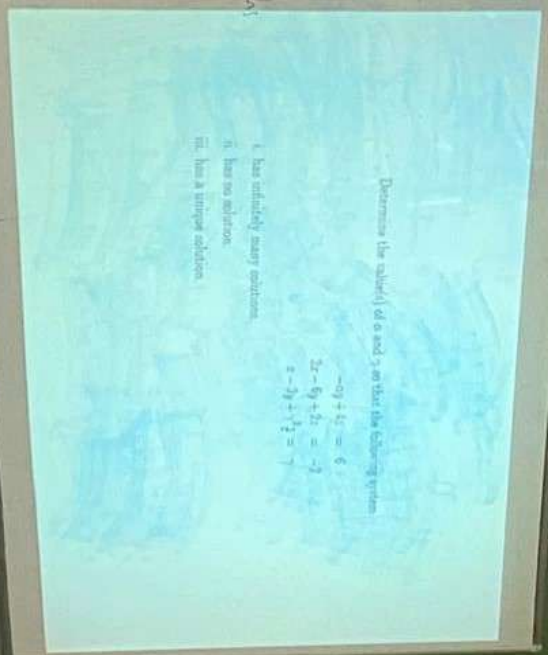
$$-y^2 = 0 \text{ and } -1-y \neq 0$$

If  $y = 1$ , then no sol.

If many sol. case

$$1-y^2 = 0 \text{ and } -1-y = 0$$

If  $y = -1$ , then inf. many sol.



$$R_1 \begin{cases} 2x_1 + 4x_2 + 6x_3 = 4 \\ x_2 - 2x_3 = -2 \\ 2x_1 + 3x_2 = 0 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 2 & 4 & 6 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-R_1+R_3 \\ R_2+R_3 \cdot R_1}} \left[ \begin{array}{ccc|c} 2 & 4 & 6 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array} \right] \xrightarrow{R_2+R_3} \left[ \begin{array}{ccc|c} 2 & 4 & 6 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array} \right]$$

unique sol.

$$\left[ \begin{array}{ccc|c} 0 & -x & y & 6 \\ 2 & -6 & 2 & -2 \\ 1 & -3 & 2 & 8 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 2 & -6 & 2 & -2 \\ 0 & -x & y & 6 \end{array} \right] \xrightarrow{\substack{-2R_1+R_2 \\ -R_1+R_3}} \left[ \begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -2 & -18 \\ 0 & -x & y & -2 \end{array} \right] \xrightarrow{1/2 R_2} \left[ \begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -1 & 9 \\ 0 & -x & y & -2 \end{array} \right] \xrightarrow{R_3+R_2} \left[ \begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & -10 \end{array} \right]$$

No sol. case  
 $-y^2 = 0$  and  $-1-y \neq 0$   
 $\Rightarrow y = 0$ , then no sol.  
 If many sol. case  
 $-y^2 = 0$  and  $-1-y = 0$  then  
 $\Rightarrow y = -1$ , then many sol.  
 Unique sol.  
 $-1-y^2 \neq 0$ , then  $y^2 + 1 = 0$

Particular case also

$$\left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

if  $x=0$ , if many sol.  
 if  $x \neq 0$ , no sol.  
 if  $x=0$ , then many sol. and particular solution is  $(0,0,0)$

$$\begin{cases} x+y+z=1 \\ x+y+z=2 \\ x+y+z=3 \end{cases} \rightarrow \begin{cases} x+y+z=1 \\ x+y+z=2 \\ x+y+z=3 \end{cases} \rightarrow \begin{cases} x+y+z=1 \\ 0=1 \\ 0=2 \end{cases}$$

1. No solution  
 2. No solution  
 3. No unique solution