The Method of Substitution

Some elementary integrals

1. $\int 1 \cdot dx = x + c$ (c: constant)
2. $\int x \cdot dx = \frac{1}{2}x^2 + c$
3. $\int x^2 \cdot dx = \frac{1}{3}x^3 + c$
4. $\int \frac{1}{x^2}\, dx = -\frac{1}{x} + c$
5. $\int \sqrt{x}\, dx = \frac{2x^{\frac{3}{2}}}{3} + c$
6. $\int \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x} + c$
7. $\int x^r\, dx = \frac{1}{r+1} x^{r+1} + c$
8. $\int \frac{1}{x}\, dx = \ln|x| + c$
9. $\int \sin(ax)\, dx = \frac{-1}{a}\cos(ax) + c$
10. $\int \cos(ax) \cdot dx = \frac{1}{a}\sin(ax) + c$
11. $\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + c$
12. $\int \csc^2 ax \cdot dx = -\frac{1}{a}\cot(ax) + c$
13. $\int \sec(ax)\tan(ax)\, dx = \frac{1}{a}\sec(ax) + c$
14. $\int \csc(ax) \cdot \cot(ax) \cdot dx = -\frac{1}{a}\csc(ax) + c$
15. $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \quad (a > 0)$
16. $\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right) + c$
17. $\int e^{ax}\, dx = \frac{1}{a} \cdot e^{ax} + c$
18. $\int b^{ax}\, dx = \frac{1}{a \ln b} b^{ax} + c$

ex:

a) $\int (x^4 - 3x^3 + 8x^2 - 6x - 7)\,dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - 3x^2 - 7x + C$

b) $\int (4\cos 5x - 5\sin 3x)\,dx = \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x + C$

c) $\int \left(\frac{1}{\pi x} + a^{\pi x}\right) dx \overset{(8)}{\underset{(18)}{=}} \frac{1}{\pi}\ln|x| + \frac{1}{\pi \cdot \ln a} \cdot a^{\pi x} + C$

In derivation

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

In integration

CR: $\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$

⟹ Using CR in integration

Let $u = g(x)$. Then $du = g'(x) \cdot dx$. Thus,

$$\int f'(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x) \cdot dx}_{du} = \int f'(u) \cdot du = f(u) + C = f(g(x)) + C$$

Some general formulas

① $\int \frac{f'(x)}{f(x)} dx \quad u=f(x)$

② $\int f'(x)\,(f(x))^n \cdot dx \quad u=f(x)$

③ $\int f'(x) \sin(f(x))\, dx \quad u=f(x)$

④ $\int f'(x)\, e^{f(x)} \cdot dx \quad u=f(x)$

⑤ $\int f'(x) \cdot \cos(f(x))\, dx \quad u=f(x)$

⑥ $\int \frac{f'(x)}{f(x)} \cdot \ln(f(x)) \cdot dx \quad u=\ln(f(x))$

ex 2:

a) $\int \frac{x}{x^2+1} dx =$ $\quad x^2+1=u \quad du=2x \cdot dx$

$x \cdot dx = \frac{du}{2}$

$\int \frac{du}{2u} = \frac{1}{2}\ln|u| + c$

b) $\int \frac{\sin(3\ln x)}{x} dx = -\frac{1}{3}\cos 3u + c \quad \ln x = u$

$du = \frac{1}{x}$

c) $\int \underbrace{\tan x \cdot \underbrace{\ln(\cos x)}_{u} \cdot dx}_{du}$

let $u = \ln(\cos x)$

$du = -\frac{\sin x}{\cos x} dx$
$\tan x$

$du = -\tan x\, dx$

$= \int u \cdot (-du) = -\int u\,du = -\frac{1}{2}u^2 + c = -\frac{1}{2}[\ln(\cos x)]^2 + c$

Theorem

Substitution in definite integral.

Suppose that g is a differentiable function on $[a,b]$ that satisfies $g(a)=A$ and $g(b)=B$.
Also, suppose that f is continuous on the range of g. Then $\int_a^b f(g(x)) \cdot g'(x)dx = \int_A^B f(u) \cdot du$ $\quad u=g(x)$

ex 3:

a) $\int_e^{e^2} \frac{dt}{t\ln t}$

Let $u=\ln t$

$du = \frac{dt}{t}$

$t=e \rightarrow u=\ln e=1$

$t=e^2 \rightarrow u \; \ln e^2=2$

$= \int_1^2 \frac{du}{u} = \ln u \Big|_1^2 = \ln 2 - \ln 1$

$= \ln 2$

b) $\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}} dx$

$u=\sqrt{x+1} \Rightarrow u=(x+1)^{\frac{1}{2}}$

$du = \frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot dx$

$\Rightarrow du = \frac{dx}{2\sqrt{x+1}}$

$\Rightarrow 2du = \frac{dx}{\sqrt{x+1}}$

$x=0 \rightarrow u=1=A$

$x=8 \rightarrow u=3=B$

$= \int_1^3 \cos u \cdot 2du = 2\int_1^3 \cos u\, du = 2\sin u \Big|_1^3$

$= 2\sin 3 - 2\sin 1$

Trignometric Integrals

1. $\int \tan x \, dx = \ln|\sec x| + c$

2. $\int \cot x \, dx = \ln|\sin x| + c$
$= -\ln|\csc x| + c$

3. $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

4. $\int \csc x \, dx = -\ln|\csc x + \cot x| + c$
$= \ln|\csc x - \cot x| + c$
1

$1 \rightarrow \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{du}{u} = -\ln|u| + c$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$= -\ln|\cos x| + c$

$= \ln|\cos x|^{-1} + c$

$= \ln|\sec x| + c$

$4 \rightarrow \ln|\csc x + \cot x|^{-1}$

$(\csc x + \cot x)^{-1} = \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)^{-1} = \left(\frac{1+\cos x}{\sin x}\right)^{-1} = \frac{\sin x}{1+\cos x} \, (1-\cos x)$

$= \frac{\sin x \, (1-\cos x)}{1-\cos^2 x \;\; \sin^2 x}$

$= \frac{1-\cos x}{\sin x} = \frac{1}{\sin x} - \frac{\cos x}{\sin x}$

$= \csc - \cot x$

$\int \sin^m x \cdot \cos^n x \, dx$ if either m or n is an odd positive integral, the integral can be done easily by substitution. If, say, $n = 2k+1$ (n is odd) where k is an integer then we can use the identity $\sin^2 x + c + \cos^2 x = 1$ to rewrite the integral in the form. (1) $\int \sin^m x \, (1-\sin^2 x)^k \cdot \cos x \cdot dx$ (2) $u = \sin x$

if m is odd we use the substitution → $u = \cos x$

$\int (1-\cos^2 x)^k \cdot \sin x \cdot \cos^n x \cdot dx$ (2')

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If the powers of sinx and cosx are even, then we can make use of double angle formulas.

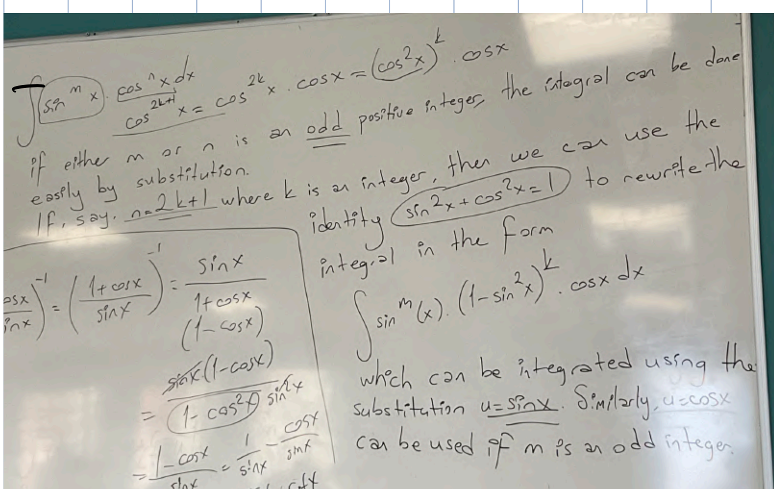
a) $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

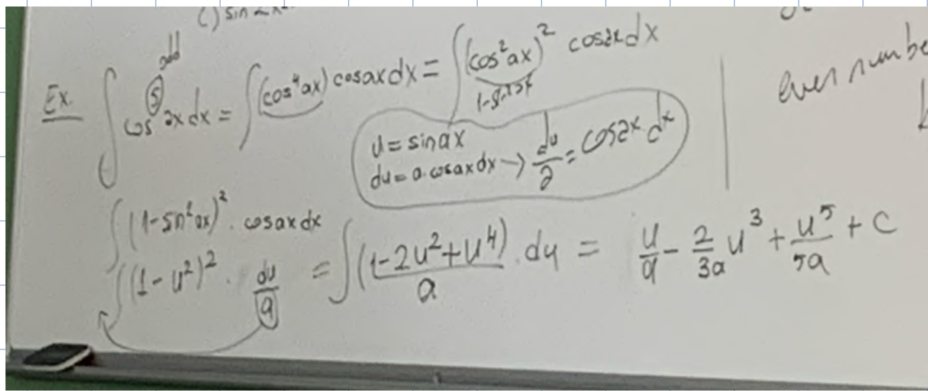
b) $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

c) $\sin^2 x = 2\sin x \cdot \cos x \Rightarrow \sin x \cdot \cos x = \frac{\sin 2x}{2}$



HW = $\int \cos^5 ax \cdot dx$



Ex. $\int \cos^{5} ax\, dx = \int (\cos^4 ax) \cos ax\, dx = \int (\cos^2 ax)^2 \cos ax\, dx$

odd

$u = \sin ax$

$du = a \cdot \cos ax\, dx \rightarrow \frac{du}{a} = \cos ax\, dx$

$\int (1-\sin^2 ax)^2 \cdot \cos ax\, dx$

$\int (1-u^2)^2 \cdot \frac{du}{a} = \int \left(\frac{1-2u^2+u^4}{a}\right) du = \frac{u}{a} - \frac{2}{3a} u^3 + \frac{u^5}{5a} + C$

even numbe

$= \frac{\sin ax}{a} - \frac{2}{3a} \sin^3 x + \frac{\sin^5 ax}{5a} + C$

Ex

$\int \sin^3 x \cdot \cos^8 x\, dx$

$= \int \sin^2 x \cdot \sin x \cos^8 x\, dx$

$\int (1-\cos^2 x) \cos^8 x \cdot \sin x\, dx$

let $u = \cos x$

$du = -\sin x\, dx$

$= -\int (1-u^2)\, u^8\, du$

$= -\int (u^8 - u^{10})\, du = -\left(\frac{u^9}{9} - \frac{u^{11}}{11}\right) = -\left(\frac{\cos^9 x}{9} - \frac{\cos^{11} x}{11}\right)$

Ex

a) $\sin^{-2/3} x \cdot \cos^3 x\, dx$

$= \int \sin^{-2/3} x \cdot \cos^2 x \cdot \cos x\, dx$

$= \int \sin^{-2/3} x \cdot (1-\sin^2 x) \cdot \cos x\, dx$

$u = \sin x \quad du = \cos x\, dx$

$= \int u^{-2/3} (1-u^2) \cdot du$

$= \int (u^{-2/3} - u^{4/3})\, du.$

using: $\int x^r \cdot dx = \frac{1}{r+1} \cdot x^{r+1} + C$

b) $\int \sin^3 x \cdot \cos^5 x \cdot dx$

$= \int \sin^2 x \cdot \sin x \cdot \cos^5 x\, dx$

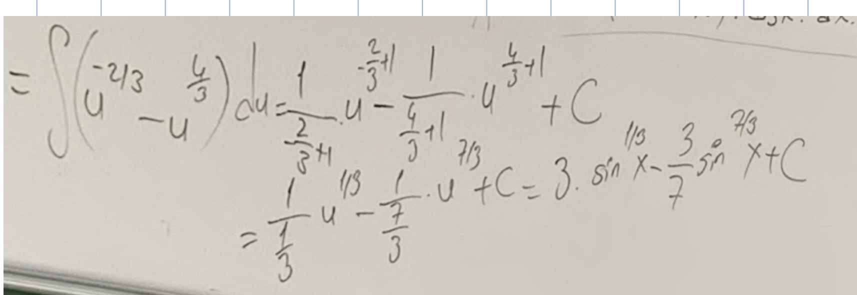
$= \int (1-\cos^2 x) \cdot \sin x \cdot \cos^5 x\, dx$

$= \int (\cos^5 x - \cos^7 x) \sin x\, dx$

$u = \cos x \quad du = -\sin x\, dx$

$= \int (u^5 - u^7) \cdot -du$

$= \int -(u^5 - u^7)\, du$



$= \int \left(u^{-2/3} - u^{\frac{4}{3}}\right) du = \frac{1}{-\frac{2}{3}+1} \cdot u^{-\frac{2}{3}+1} - \frac{1}{\frac{4}{3}+1} \cdot u^{\frac{4}{3}+1} + C$

$= \frac{1}{\frac{1}{3}} u^{1/3} - \frac{1}{\frac{7}{3}} \cdot u^{7/3} + C = 3 \cdot \sin^{1/3} x - \frac{3}{7} \sin^{7/3} x + C$

$$\int \cos ax\,dx = \frac{1}{2}\sin ax + C$$

Ex $\int \cos^2 x\,dx$

$$= \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

Ex $\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$$= \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)\,dx$$

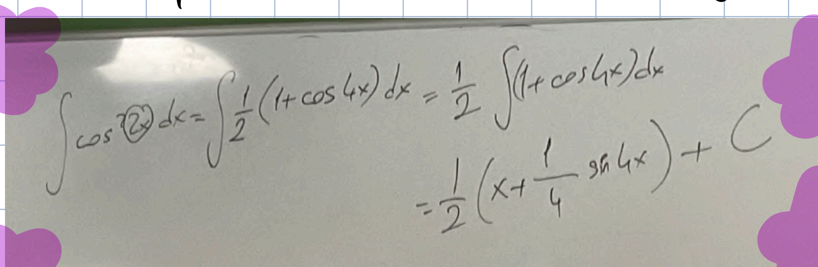
$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x)\,dx$$

$$= \frac{x}{4} - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1}{2}(1+\cos 4x)\,dx$$

$$= \frac{x}{4} - \frac{\sin^2 x}{4} + \frac{1}{4}\cdot\frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C$$

$$= -\frac{\sin 2x}{4} + \frac{3x}{8} + \frac{\sin 4x}{32} + C$$



Ex $\int \sin^2 x \cos^2 x \, dx$

$= \int (\sin x \cdot \cos x)^2 dx$

$= \int \left(\frac{\sin 2x}{2}\right)^2 dx$

$= \int \frac{\sin^2 2x}{4} dx$

$= \frac{1}{4} \int \sin^2 2x \, dx$

$= \frac{1}{2} (1 - \cos 4x) dx$

$= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x\right) + C$

$= \frac{x}{8} - \frac{\sin 4x}{32} + C$

a)

b) $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

c) $\frac{\sin 2x}{2} = \sin x \cos x$

Using the identities $\sec^2 x = 1 + \tan^2 x$ and $\csc^2 x = \frac{1}{2} + \cot^2 x$, one of the substitutions $u = \sec x$, $u = \tan x$, $u = \csc x$ or $u = \cot x$, we can evaluate integrals of the form

$\int \sec^m x \cdot \tan^n x \, dx$ or $\int \csc^m x \cdot \cot^n x \, dx$.

unless m is odd and n is even

Ex a) $\int \underbrace{\tan^2 x}_{\sec^2 x - 1} dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

b) $\int \sec^4 t \, dt = \int (1 + \tan^2 x) \cdot \sec^2 x \, dx$ $\tan x = u$, $\sec^2 x \, dx = du$

$= \int (1 + u) \, du$

$= u + \frac{u^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C$

Ex $\int \sec^3 x \tan^3 x \, dx$

$u = \sec x$

$du = \tan x \sec x \, dx$

$= \int \sec^2 (\sec^2 - 1) \tan x \sec \, dx$

$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du = \left(\frac{\sec^5 x}{5} \right) - \frac{(\sec^3 x)}{3} + C$

$= \frac{u^5}{5} - \frac{u^3}{3}$

Ex $\int e^{2x} \sin(e^{2x}) \, dx = \frac{1}{2} \int \sin u \, du = \frac{1}{2} (-\cos u) + C$

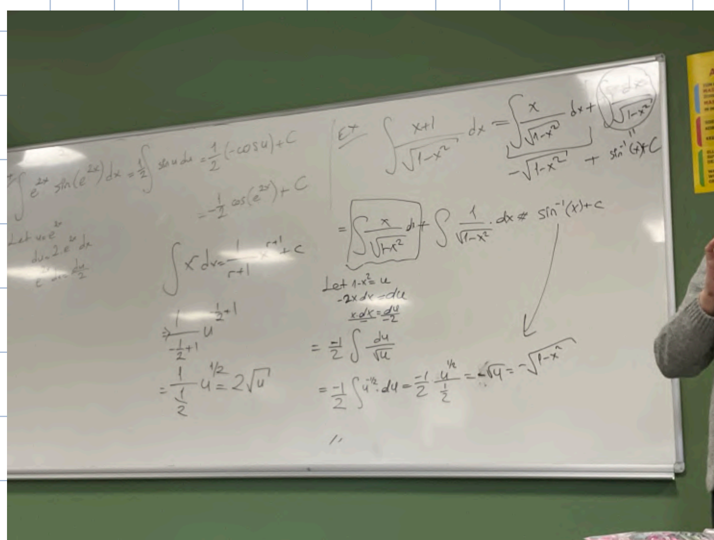
let $u = e^{2x}$

$du = 2e^{2x} \, dx$

$e^{2x} dx = \frac{du}{2}$

$= -\frac{1}{2} \cos(e^{2x}) + C$

Ex $\int \frac{x+1}{\sqrt{1-x^2}} \, dx$



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Integration by Parts

$$\int u\,dv = u\cdot v - \int v\,du$$

Log functions come before. LIATE — exp, trigo, algebraic poly, inverse, log

Ex: $\int_1^e x^3(\ln x)^2\,dx =$

Solution

Let $u=(\ln x)^2$, $dv = x^3 dx$

$du = 2\ln x \frac{1}{x} dx$

$du = \frac{2\ln x\,dx}{x}$ $\quad v = \frac{x^4}{4}$

$$(\ln x)^2\cdot\frac{x^4}{4}\Big|_1^e - \int_1^e \frac{x^4}{\cancel{4}_2}\cdot\frac{\cancel{2}\ln x}{x}dx$$

$$= (\ln e)^2\frac{e^4}{4} - (\ln 1)^2\cdot\frac{1^4}{4} - \frac{1}{2}\int_1^e x^3\ln x\,dx$$

$$= \frac{e^4}{4} - \frac{1}{2}\int_1^e x^3\ln x\,dx$$

Let $u=\ln x$, $dv = x^3dx$

$du = \frac{dx}{x}$

$$= \frac{e^4}{4} - \frac{1}{2}\left(\ln x\cdot\frac{x^4}{4}\Big|_1^e - \int_1^e \frac{x^{\cancel{4}\,3}}{4}\cdot\frac{dx}{\cancel{x}}\right)$$

$$\frac{x^4}{4}\Big|_1^e$$

$$= \frac{e^4}{4} - \frac{1}{2}\left(\frac{e^4}{4} - \frac{1}{4}\left(\frac{e^4}{4}-\frac{1^4}{4}\right)\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\left(\frac{e^4}{4}-\frac{1}{4}\right) = \frac{e^4}{4} - \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32}$$

$$= \frac{8e^4 - 4e^4 + e^4 - 1}{32} = \frac{5e^4-1}{32}$$

ex

$$\int_0^1 \sqrt{x}\,\sin(\pi\sqrt{x})\,dx$$

① Method of substitution

② Integration by parts

Let $x = t^2 \to (\sqrt{x} = t)$

$dx = 2t\,dt$ change boundries $(0-1)$

$x = 0 \Rightarrow 0 = t^2 \Rightarrow t = 0$

$x = 1 \Rightarrow 1 = t^2 \Rightarrow t = 1$

$$= \int_0^1 \sqrt{x}\sin(\pi\sqrt{x})\,dx$$

$$\overset{①}{=} \int_0^1 t\sin(\pi t)\cdot 2t\,dt = 2\int_0^1 t^2 \sin(\pi t)\,dt \overset{②}{=} 2\left(t^2\cdot -\frac{\cos(\pi t)}{\pi}\Big|_0^1 - \int_0^1 -\frac{\cos(\pi t)}{\pi}\cdot 2t\,dt\right)$$

Let $u = t^2$, $dv = \sin(\pi t)dt$

$du = 2t\,dt$ $\quad v = -\frac{\cos(\pi t)}{\pi}$

DONT change boundries

$$= 2\left(\underline{1^2(-\cos\pi)} + \frac{2}{\pi}\int_0^1 \cos(\pi t)\cdot t\,dt\right)$$

integration by parts

$$= 2\left(\frac{1}{\pi} + \frac{2}{\pi}\int_0^1 \cos(\pi t)\cdot t\,dt\right) \overset{③}{=} 2\left(\frac{1}{\pi} + \frac{2}{\pi}\left(-\frac{2}{\pi^2}\right)\right) = \frac{2}{\pi} - \frac{8}{\pi^3} \text{ answer}$$

$$③ \int_0^1 \cos(\pi t)\cdot t\,dt = t\cdot\frac{\sin(\pi t)}{\pi}\Big|_0^1 - \int_0^1 \frac{\sin(\pi t)}{\pi}\,dt = -\frac{1}{\pi}\underbrace{\int_0^1 \sin(\pi t)\,dt}_{} = \frac{1}{\pi}\left(\frac{\cos(\pi t)}{\pi}\Big|_0^1\right)$$

Let $u = t$, $dv = \cos(\pi t)dt$

$du = dt$ $\quad v = \frac{\sin(\pi t)}{\pi}$

$$-\frac{\cos(\pi t)}{\pi}\Big|_0^1 \qquad = \frac{1}{\pi}\left(\frac{-1}{\pi} - \frac{1}{\pi}\right) = \frac{1}{\pi}\cdot\frac{-2}{\pi} = -\frac{2}{\pi^2}$$

ex $\int \sin x \cdot \ln(\cos x)\,dx$

Solution

Let $u = \ln(\cos x)$, $dv = \sin x\,dx$

$du = \frac{-\sin x}{\cos x}dx$, $v = -\cos x$

$$\int \sin x \ln(\cos x)\,dx = \ln(\cos x)(-\cos x) - \int(-\cancel{\cos x})\cdot\frac{-\sin x}{\cancel{\cos x}}\,dx$$

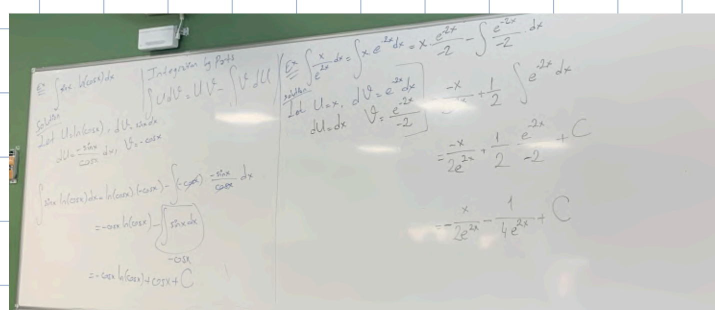
$$= -\cos x\ln(\cos x) - \int \sin x\,dx \quad -\cos x$$

$$= -\cos x\ln(\cos x) + \cos x + C$$

ex $\int \frac{x}{e^{2x}}\,dx$

$\int x\cdot e^{-2x}\,dx$

let $u = x$, $dv = e^{-2x}dx$



Integrals of Rational Functions

Polynominals

$$\frac{p(x)}{q(x)} \qquad \int \frac{P(x)}{Q(x)}\,dx$$

$(Q(x) \neq 0)$ where P and Q are polinomials. Recall that a polynominal is a function P of the form.

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ where n is a non-negative integer, $a_0, a_1, \dots, a_n$ are constants and $a_n \neq 0$. We call n the degree of P.

A quotient $\frac{p(x)}{Q(x)}$ is called a rational function.

Ex

$$\int \frac{x^3+3x^2}{x^2+1}\,dx = \int x+3 - \frac{x+3}{x^2+1}\,dx = \int \left(x+3 - \frac{x}{x^2-1} - \frac{3}{x^2+1}\right)dx$$

$$= \int (x+3)\,dx - \int \frac{x}{x^2-1}\,dx - \int \frac{3}{x^2+1}\,dx$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2}\ln(x^2+1) - 3.\tan^{-1}x + C$$

P: x^3+3x^2, Q: x^2+1

$$\begin{array}{l|l} x^3+3x^2 & x^2+1 \\ \hline -x^3+x & x+3 \\ 3x^2-x & \\ 3x^2+3 & \\ -x-3 & \end{array}$$

$-x-3 = -(x+3)$

Ex

$$\int \frac{x}{2x-1}\,dx = \int \frac{1}{2}\left(1+\frac{1}{2x-1}\right)dx = \frac{1}{2}\int \left(1+\frac{1}{2x-1}\right)dx = \frac{1}{2}\left(x+\frac{1}{2}\ln|2x-1|\right)+C$$

$$= \frac{1}{2} \cdot \frac{2x}{2x-1} = \frac{1}{2} \cdot \frac{(2x-1+1)}{2x-1} = \frac{1}{2}\left(\frac{2x-1}{2x-1} + \frac{1}{2x-1}\right) = \frac{1}{2}\left(1+\frac{1}{2x-1}\right)$$

Linear and Quadratic Denominators

Suppose that $Q(x)$ has degree 1. Thus, $Q(x) = ax+b$, where $a \neq 0$. Then $P(x)$ must have degree 0 and be a constant c. We have $\frac{P(x)}{Q(x)} = \frac{c}{ax+b}$. The substitution $u = ax+b$ leads to,

$du = a\,dx$

$dx = \frac{du}{a}$

$$\int \frac{c}{ax+b}\,dx = \int \frac{c}{u} \cdot \frac{du}{a} = \frac{c}{a}\int \frac{du}{u} = \frac{c}{a}\ln|u| + C = \frac{c}{a}\ln|ax+b| + C$$

If $c=1$ $\quad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

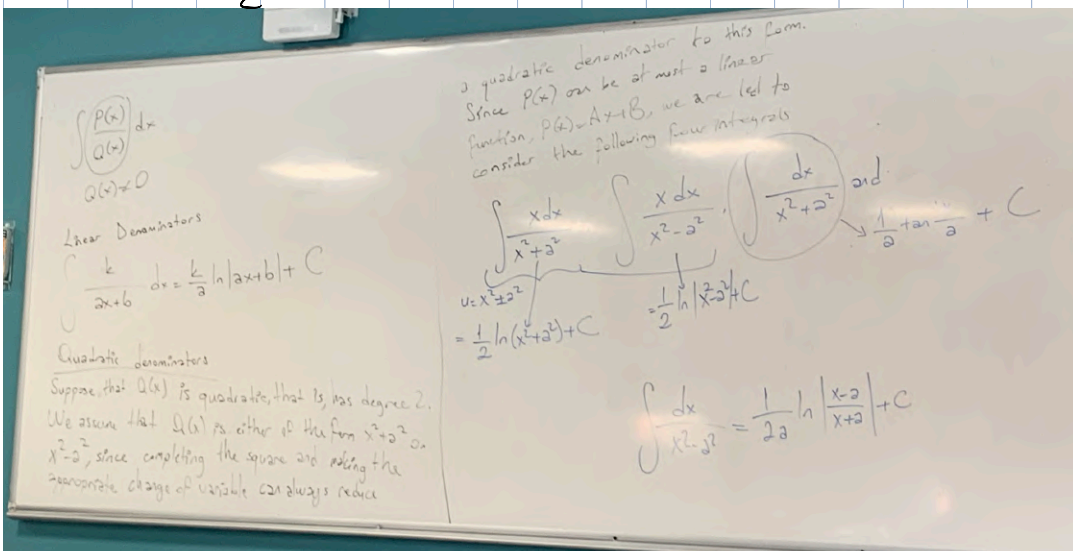
ex $\displaystyle\int \frac{5}{4x+7}\,dx$

$u = 4x+7$
$du = 4\,dx$
$dx = \frac{du}{4}$

$$= \int \frac{5}{u}\,\frac{du}{4} = 5\int \frac{du}{4} = \frac{5}{4}\ln|4x+7| + C$$

ex $$\int \frac{1}{7x-9}\,dx = \frac{1}{7}\ln|7x-9| + C$$

ex $$\int \frac{1}{\frac{3}{2}x+\frac{5}{2}} = \frac{2}{3}\ln\left|\frac{3}{2}x+\frac{5}{2}\right| + C$$



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To obtain the last formula, we use the integrand as a sum of two fractions with linear denominators.

$$\frac{1}{\underbrace{x^2-a^2}_{(x-a)(x+a)}} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)} = \frac{Ax+Aa+Bx-Ba}{x^2-a^2}$$

$(x+a)$ $(x-a)$

$$\boxed{= \frac{x(A+B)+Aa-Ba}{x^2-a^2}}$$

$$1 = x\underbrace{(A+B)}_{0} + \underbrace{Aa-Ba}_{1}$$

$A+B=0 \Rightarrow B=-A$
$Aa-Ba=1$ $\quad -2Ba=1$
$Aa+Aa=1$ $\quad B=-\frac{1}{2a}$
$2Aa=1$
$A=\frac{1}{2a}$

$$\int \frac{dx}{x^2-a^2} = \int\left(\frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}\right)dx = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right] = \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

$$= \frac{1}{2a}\ln\frac{|x-a|}{|x+a|} + C$$

Partial fractions

Suppose that a polynomial $Q(x)$ is of degree n and that it's highest degree term is x^n (coefficient is 1) suppose that Q factors into a product of n distinct linear (degree 1) factors, say $Q(x) = (x-a_1)(x-a_2)\dots(x-a_n)$ where $a_i \neq a_j$ if $i \neq j$ $1 \leq i, j \leq n$. If $\deg P(x) < n = \deg Q(x)$, then $\frac{P(x)}{Q(x)}$ has a partial fraction decomposition of the form.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$$ for certain values of the constants $A_1, \dots, A_n$.

#1

The first method is to add up the fractions in the decomposition, obtaining a new fraction $\frac{S(x)}{Q(x)}$, a polynomial of degree one less than that of $Q(x)$.

The constant $A_1, \dots A_n$ are determined by solving the n linear functions resulting from equating the coefficient of like powers of x in two polynomials S and P.

#2

The second method depends on following the second observation:

If we multiply the p.f.d by $x-a_j$, we get

$$(x-a_j)\frac{P(x)}{Q(x)} = A_1 \frac{x-a_j}{x-a_1} + A_2 \frac{x-a_j}{x-a_2} + \dots A_{j-1}\frac{x-a_j}{x-a_{j-1}} + A_j + \dots + A_n \frac{x-a_j}{x-a_n}$$

All terms on the right side are 0 at $x=a_j$ except the j^{th} term, A_j.

Hence $A_j = \lim_{x \to a_j} (x-a_j)\frac{P(x)}{Q(x)} = \frac{P(a_j)}{(a_j-a_1)(a_j-a_2)\cdots(a_j-a_n)}$

EX $\int \frac{x+4}{x^2-5x+6}dx \Rightarrow \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2}_{(x-3)} + \frac{B}{x-3}_{(x-2)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$1x+4 = x\underbrace{(A+B)}_{1} - 3A-2B$ 4

$2/ A+B=1$

$-3A-2B=4$

$2A+2B=2$

$+ \; -3A-2B=4$

$-A=6$

$A=-6$

$$= \int \left(\frac{-6}{x-2} + \frac{7}{x-3} \right) dx$$

$$= -6\int \frac{dx}{x-2} + 7\int \frac{dx}{x-3}$$

$$= -6\ln|x-2| + 7\ln|x-3| + C$$

Ex $\displaystyle\int \frac{x^3+2}{x^3-x}\,dx = \int 1+\frac{x+2}{x^3-x}\,dx = \int \left(1-\frac{2}{x}+\frac{3}{2(x-1)}+\frac{1}{2(x+1)}\right)dx = x-2\ln|x|+\frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$

$$\frac{x^3+2}{x^3-x} = \frac{x^3-x+x+2}{x^3-x} = \frac{x^3-x}{x^3-x} + \frac{x+2}{x^3-x} = 1+\frac{x+2}{x^3-x} = 1-\frac{2}{x}+\frac{3}{2(x-1)}+\frac{1}{2(x+1)}$$

$$\frac{x+2}{\underbrace{x^3-x}_{x(x-1)(x+1)}} = \frac{A}{x}_{(x^2-1)} + \frac{B}{x-1}_{(x^2+x)} + \frac{C}{x+1}_{(x^2-x)} = \frac{Ax^2-A+Bx^2+Bx+Cx^2-Cx}{x(x-1)(x+1)} = \frac{x^2(A+B+C)+x(B-C)-A}{x^3-x}$$

$$x^2\underbrace{(A+B+C)}_{0} + x\underbrace{(B-C)}_{1} - A = 1x+2$$

$-A=2 \Rightarrow A=-2$ //

$A+B+C=0$

$-2+B+C=0$

$B+C=2$

$+\ B-C=1$

$2B=3$

$B=3/2$ //

$\frac{3}{2}+C=2$

$C=\frac{1}{2}$ //

$$-\frac{2}{x}+\frac{3}{2(x-1)}+\frac{1}{2(x+1)}$$

Ex $\displaystyle\int \frac{2+3x+x^2}{x(x^2+1)}\,dx$ odd

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x}_{(x^2+1)} + \frac{Bx+C}{x^2+1}_{(x)}$$

$$= \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

$$= \frac{x^2(A+B)+Cx+A}{x(x^2+1)}$$

must be same

$A+B=1$

$C=3$ //

$A=2$ //

$B=-1$ //

$$\frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{2}{x}-\frac{x-3}{x^2+1}$$

$$\int \frac{2+3x+x^2}{x(x^2+1)}dx = \int\left(\frac{2}{x}-\frac{x-3}{x^2+1}\right)dx = 2\int\frac{dx}{x} - \int\frac{x\,dx}{x^2+1} + 3\int\frac{dx}{x^2+1}$$

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C$$

Completing the Square

$$Ax^2+Bx+C = A\left(x^2+\frac{B}{A}x+\frac{C}{A}\right)$$

$$= A\left(x^2+\frac{B}{A}x+\frac{B^2}{4A^2}+\underbrace{\frac{C}{A}-\frac{B^2}{4A^2}}_{(4A)}\right)$$

$$= A\left(x+\frac{B}{2A}\right)^2 + \frac{4AC-B^2}{4A}$$

$$\boxed{u = x+\frac{B}{2A}}$$

Ex $\int \frac{1}{x^3+1}\,dx$

Sol $Q(x) = x^3+1 = (x+1)(x^2-x+1)$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1}\,{}_{(x^2-x+1)} + \frac{Bx+C}{x^2-x+1}\,{}_{(x+1)} = \frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{(x^2-x+1)(x+1)}$$

$$= \frac{x^2(A+B)+x(-A+B+C)+A+C}{x^3+1}$$

$A+B=0$

$-A+B+C=0 \qquad B=-A$

$A+C=1 \qquad A=1/3$

$-2A+C=0. \qquad C=2/3$

$-3A=-1 \qquad B=-1/3$

$A=1/3$

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)} \Rightarrow \int \frac{1}{x^3+1}\,dx = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x-2}{x^2-x+1}\,dx$$

$$x^2-x+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int\frac{1}{x^3+1}\,dx = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x^2}{x^2-x+1}\,dx$$

let $u = x-\frac{1}{2}$

$du = dx$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{x-\frac{1}{2}-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{u-\frac{3}{2}}{u^2+3/4}\,du$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\cdot\frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{3}\,\frac{2}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{3}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+C$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

Ex $\int \frac{1}{x(x-1)^2}\,dx = \int \frac{dx}{x} - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$

$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$

$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C$

$$\frac{1}{\underbrace{x(x-1)^2}_{x(x-1)(x-1)}} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{(x-1)^2}$$

$(x-1)^2 \quad (x(x-1)) \quad (x)$

$$= \frac{A(x^2-2x+1) + Bx(x-1) + Cx^2 + Dx}{x(x-1)^2}$$

$$= \frac{x^2(A+B+C) + x(-2A-B+D) + A}{x(x-1)^2}$$

$A+B+C=0$

$-2A-B+D=0$

$A=1$

$\frac{1}{\quad}$

$-2-B+D=0$

$B+C=-1$

$D-B=2$

$D+C=3$

$B=-1$

$C=1$

$A=1$

$\ln \frac{P(x)}{Q(x)}$

$= \deg P(x) = \deg Q(x) - 1$

24 march 2022

Ex $\int \frac{x^2+2}{4x^5+4x^3+x}\,dx \qquad \frac{x^2+2}{4x^5+4x^3+x} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$

a) $4x^5+4x^3+x = x(4x^4+4x^2+1)$

$= x(2x+1)^2$

$$= \frac{A(4x^4+4x^2+1) + (Bx^2+Cx)(2x^2+1) + Dx^2 + Ex}{x(2x^2+1)^2}$$

$$= \frac{x^4(4A+2B) + 2Cx^3 + x^2(4A+B+D) + x(C+E) + A}{x(2x^2+1)^2}$$

$= x^2+2 = x^4(4A+2B) + 2Cx^3 + x^2(4A+B+D) + x(C+E) + A$

$4A+2B=0 \quad B-D=0 \quad A=2 \quad B=-4 \quad C=0 \quad D=-3 \quad E=0$

$2C=0 \quad C+E=0$

$4A+B+D=0 \quad A=2$

$$\frac{x^2+2}{x(2x^2+1)} = \frac{2}{x} - \frac{4x}{2x^2+1} - \frac{3x}{(2x^2+1)^2}$$

$$\int \frac{x^2+2}{x^5+4x^3+x} = 2\int \frac{dx}{x} - 4\int \frac{x\,dx}{2x^2+1} - 3\int \frac{x\,dx}{(2x^2+1)^2} = 2\ln|x| - 4\int \frac{du}{4u} - 3\int \frac{du}{4u^2}$$

$$= 2\ln|x| - \frac{4}{4}\int \frac{du}{u} - \frac{3}{4}\int u^{-2}\,du$$

$$= 2\ln|x| - \ln|u| - \frac{3}{4}\left(-\frac{1}{u}\right) + C$$

$$2\ln|x| - \ln|2x^2+1| + \frac{3}{4(2x^2+1)} + C$$

$$\frac{3x+2}{x^3-2x^2} = \frac{3x+2}{\underbrace{x^2(x-2)}_{x.x.(x-2)}} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$

$A + C = 0$

$-2A + B = 3$

$-2B = 2$

$B = -1$

$A = -2$

$C = 2$

$$= \frac{3x+2}{x^2(x-2)} = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-2}$$

$$\int \frac{3x+2}{x^2(x-2)}dx = -2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-2} = -2\ln|x| - \left(-\frac{1}{x}\right) + 2\ln|x-2| + C$$

$$= \frac{1}{x} + 2\left(\ln|x-2| - \ln|x|\right) + C \quad \ln\left|\frac{x-2}{x}\right|$$

$$\boxed{= \frac{1}{x} + 2\ln\left|\frac{x-2}{x}\right| + C}$$

Inverse Functions

a) Inverse Trigonometric Functions

Three very useful inverse substitutions are:

- $x = a\sin\theta$
- $x = a\tan\theta$
- $x = a\sec\theta$

* These correspond to direct substitution or method of substitution

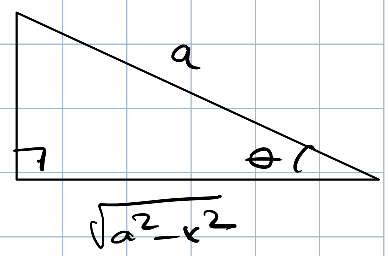
- $\theta = \sin^{-1}\frac{x}{a}$
- $\theta = \sec^{-1}\frac{x}{a}$
- $\theta = \tan^{-1}\frac{x}{a}$

$= \cos^{-1}a$

① Inverse sine substitution. Integrals involving $\sqrt{a^2-x^2}$ $(a>0)$ can frequently be reduced to a simpler form by means of the substitution.

$$x = a\sin\theta \text{ , or, } \theta = \sin^{-1}\frac{x}{a}$$

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}} = a\cos\theta$$



$$\Rightarrow \cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{x}{a}}{\frac{\sqrt{a^2-x^2}}{a}} = \frac{x}{\cancel{a}} \cdot \frac{\cancel{a}}{\sqrt{a^2-x^2}} = \frac{x}{\sqrt{a^2-x^2}}$$

Ex $\displaystyle\int \frac{1}{(5-x^2)^{3/2}}\,dx$ $\qquad (5-x^2)^{\frac{3}{2}} = \sqrt{(5-x^2)^3}$ $\qquad \left(\begin{array}{l} \text{let } x = \sqrt{5}\sin\theta \\ dx = \sqrt{5}\cos\theta\, d\theta \end{array}\right)$

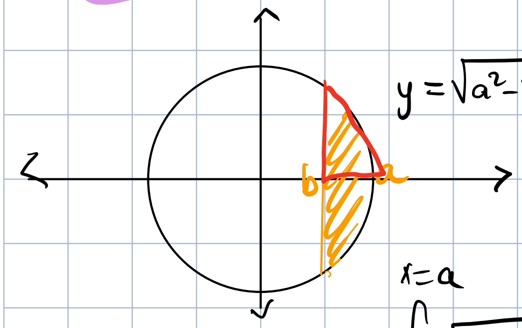
$$\int \frac{1}{(5-x^2)^{3/2}}\,dx = \int \frac{\sqrt{5}\cos\theta\, d\theta}{(\underbrace{5-5\sin^2\theta}_{5\cdot\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}})^{3/2}} = \int \frac{\sqrt{5}\cos\theta\, d\theta}{(5.\cos^2\theta)^{3/2}}$$

$$= \int \frac{\sqrt{5}\cos\theta\, d\theta}{5^{3/2}\cdot\cos^3\theta} = \int \frac{d\theta}{5\cos^2\theta} = \frac{1}{5}\underbrace{\int \sec^2\theta\, d\theta}_{\tan\theta}$$

$$= \frac{1}{5}\tan\theta + C$$

$$\boxed{= \frac{x}{5\sqrt{5-x^2}} + C}$$

Ex Find the area of circular segment shaded in figure.



$$TA = 2\int_b^a \sqrt{a^2-x^2}\,dx$$

Let $x = a\sin\theta$ $dx = a\cos\theta\, d\theta$

$$2\int_{x=b}^{x=a}\sqrt{a^2-x^2}\,dx = 2\int_{x=b}^{x=a}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

$$= 2\int_{x=b}^{x=a} a^2\cos^2\theta\, d\theta$$

$\sin\theta = \frac{x}{a}$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$$= a^2(\theta + \sin\theta\cos\theta)\Big|_{x=b}^{x=a}$$

$$= a^2\left(\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2}\right)\Big|_b^a$$

$$= \frac{\pi}{2}a^2 - a^2\sin^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} \;//$$

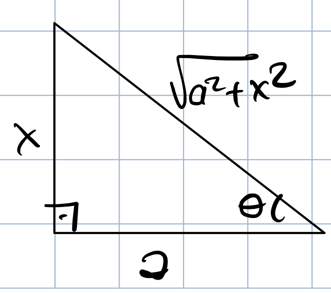
The Inverse Tangent Substitution

Integrals involving $\sqrt{a^2+x^2}$ or $\frac{1}{x^2+a^2}$ $(a>0)$ are often simplified by the substitution:

$x = a\tan\theta$, or, $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}} = a\sec\theta$$

$$\Rightarrow \sin\theta = \frac{x}{\sqrt{a^2+x^2}},\ \cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$



Ex

a) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{\cancel{2}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{2\sec\theta}}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left|\sqrt{4+x^2} + x\right| - \ln 2 + C \quad C_1$$

$$= \ln\left|x + \sqrt{4+x^2}\right| + C_1$$

b) $\displaystyle\int \frac{dx}{(1+9x^2)^2} = \int \frac{dx}{(1+(3x)^2)^2}$

Let $3x = \tan\theta \Rightarrow 1+9x^2 = \underbrace{\tan^2\theta + 1}_{\sec^2\theta}$

$3dx = \sec^2\theta\, d\theta$

$$\int \frac{dx}{(1+9x^2)^2} = \int \frac{\sec^2\theta\, d\theta}{3\sec\theta}$$

$$= \frac{1}{3}\int \frac{\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{1}{3}\int \cos^2\theta\, d\theta$$

$1 + 9x^2 = \sec^2\theta$

$\theta = \tan^{-1} 3x$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{1}{3}\int \frac{(\cos 2\theta) + 1}{2}\, d\theta$$

$$= \frac{1}{3}\cdot\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{6}(\theta + \sin\theta\cos\theta) + C$$

$$\boxed{\sin\theta\cos\theta = \frac{ax}{a^2+x^2}}$$

$$= \frac{1}{6}\left(\tan^{-1}(3x) + \frac{3x}{1+9x^2}\right) + C$$