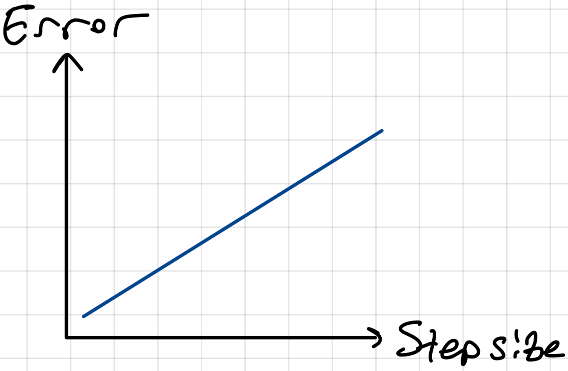
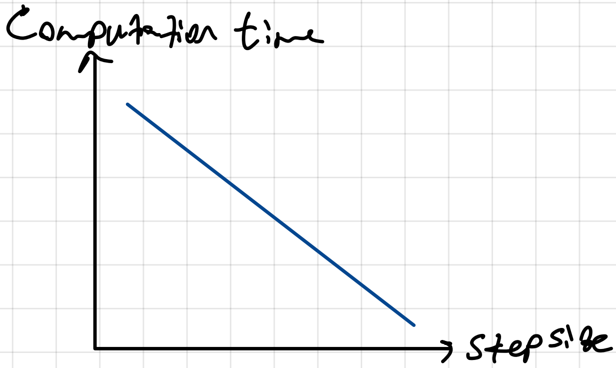


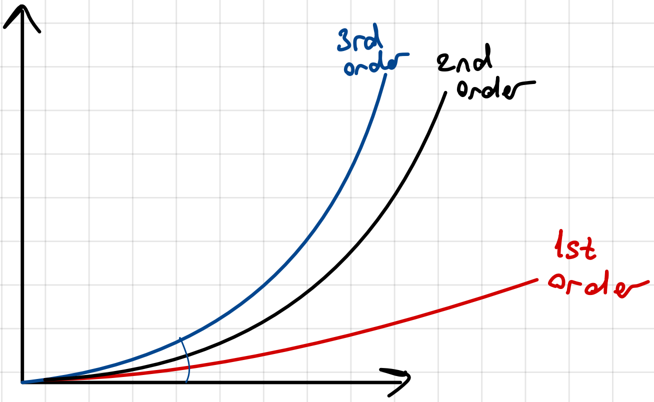


0th order approx.   1th order approx.   2nd   3

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$$

It's not ideal to use first order approx. (constant)

$$\text{Truncation Error} = e^x - \left[1 + x + \frac{x^2}{2!}\right]$$



Numerical Iterations

Convergent

Go a solution
⇓
at the end of the Iteration you get correct result

Divergent

going far from final correct result

Numerical Integration

Analytical Function
Can not be integrated analytically

Data
- 1D
- 2D

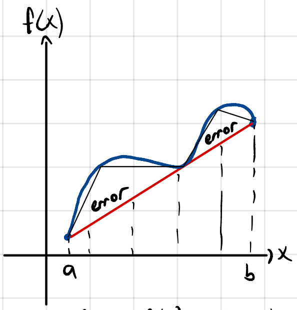
t	$a(t)$

$a = \frac{dv}{dt}$

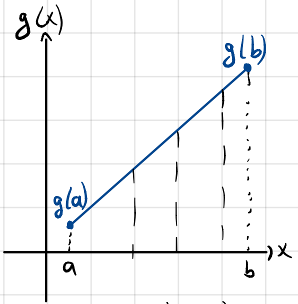
$v(t) = ?$ $\quad v = \underbrace{\int a\,dt}_{\text{numerical}}$

Trapeziodal Rule $y = ax+b$

1st order approx.



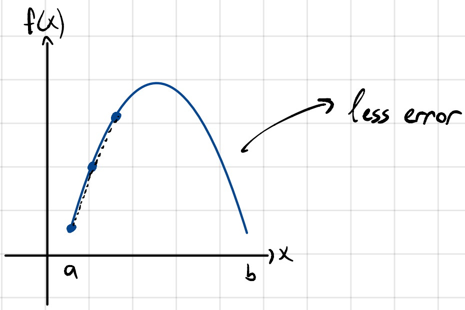
$$\frac{f(a) + f(b)}{2} \cdot (b-a)$$

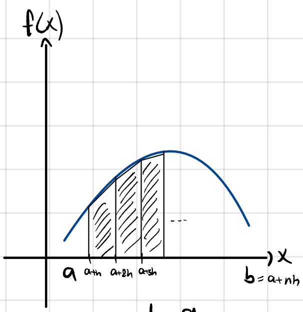


$$I = \frac{g(a) + g(b)}{2} \cdot (b-a)$$

Simpson's Rule $y = ax^2+bx+c$

2nd order approx





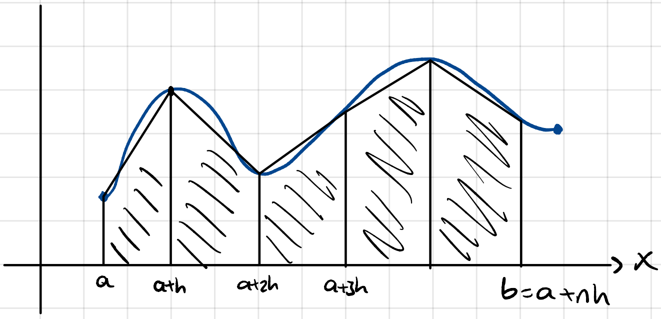
$$h = \frac{b-a}{n}$$

n: 10

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(x_n)$$

1st D.D.

$$\frac{\dfrac{f(x_{n+2}) - f(x_{n+1})}{x_{n+2} - x_{n+1}} - \dfrac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}}{x_{n+2} - x_n} = f''(n)$$



$a \to a+ih$

$$\int_a^b f(x)\,dx \overset{\sim}{=} \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \int_{a+2h}^{a+3h} f(x)dx$$

$$+ \cdots\cdots \int_{a+(n-1)h}^{a+nh=b} f(x)\,dx$$

Interpolation

1) Direct Method

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad , \quad x_0 \le x \le x_1$$

NDDP: Example:

Determine the interpolating polynomial for the follwing data:

x	2	4	5	7
f(x)	5	-5	-40	10

$$\frac{y_1 - y_0}{x_1 - x_0}$$

Solution:

x_k	$f[\,] = f()$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
2	5			
		$\frac{-5-5}{4-2} = -5$		
4	-5		$\frac{-35-(-5)}{5-2} = -10$	
		$\frac{-40-(-5)}{5-4} = -35$		$\frac{20-(-10)}{7-2} = 6$
5	-40		$\frac{25-(-35)}{7-4} = 20$	
		$\frac{10-(-40)}{7-5} = 25$		
7	10			

$$p(x) = 5 - 5(x-2) - 10(x-2)(x-4) + 6(x-2)(x-4)(x-5)$$

Given the data

x_i	$f(x_i)$
1	3
2	6
3	19
5	99

Calculate $f(4)$ using Newton's interpolating polynomials of order 1 through 3.

x_i	$f[x_i]$	First	Second	Third
1	3	3	5	1
2	6	13	9	
3	19	40		
5	99			

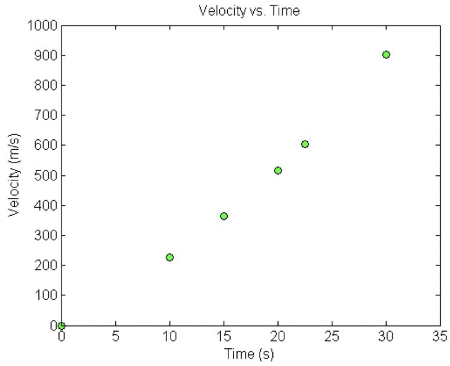
$$P_1(x) = 3 + 3(x-1)$$

$$P_2(x) = 3 + 3(x-1) + 5(x-1)(x-2)$$

$$P_3(x) = 3 + 3(x-1) + 5(x-1)(x-2) + 1(x-1)(x-2)(x-3)$$

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Lagrangian method for quadratic interpolation.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



$t_0 = 15$ $\quad V_0 = 362.78$

$t_1 = 20$ $\quad V_1 = 517.35$

$t_2 = 10$ $\quad V_2 = 227.04$

$P_1(x) = f(x_0)\,l_0 + f(x_1)\,l_1(x)$

$P_1 =$

$P_3(t) = 362.78\left(\frac{4}{5}\cdot\frac{6}{5}\right) + 517.35\left(\frac{1}{5}\cdot\frac{3}{5}\right) + 227.04\left(-\frac{1}{5}\ \frac{2}{5}\right)$

$$l_0 = \frac{t-t_1}{t_0-t_1}\ \frac{t-t_2}{t_0-t_2}$$

$$l_1 = \frac{t-t_0}{t_1-t_0}\ \frac{t-t_2}{t_1-t_2}$$

$\frac{-4}{-10} = \frac{2}{5}$

$$l_2 = \frac{t-t_0}{t_2-t_0}\ \frac{t-t_1}{t_2-t_1}$$

$\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)$

$\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)$

x	$f(x)$
0	3
2	60
4	90
10	120

f(8) = ?

$l_0 = \frac{8-2}{-2}\ \frac{8-4}{-4}\ \frac{8-10}{-10}$, $\quad l_0 = \frac{3}{5}$

$l_1 = \frac{8-0}{2-0}\ \frac{8-4}{2-4}\ \frac{8-10}{2-10}$ $\quad l_1 = -2$

$l_2 = \frac{8-0}{4-0}\ \frac{8-2}{4-2}\ \frac{8-10}{4-10}$ $\quad l_2 = 2$

$l_3 = \frac{8-0}{10-0}\ \frac{8-2}{10-2}\ \frac{8-4}{10-4}$ $\quad l_3 = \frac{2}{5}$

Least Square Fitting

$$\boxed{y = a_0 + a_1 X}$$

$$\boxed{a_1 = \frac{n \cdot \sum (x_i y_i) - \sum x_i \sum y_i}{n \cdot \sum x_i^2 - (\sum x_i)^2}}$$

$$\boxed{a_0 = \bar{y} - a_1 \bar{x}}$$

Example 1: Use least-squares regression to fit a straight line to

x	1	3	5	7	10	12	13	16	18	20
y	4	5	6	5	8	7	6	9	12	11

$n = 10$

$\sum x_i = 105$ $\quad \sum y_i = 73$ $\quad \sum x y_i = 906$ $\quad \sum x_i^2 = 1477$

$\bar{x} = 10.5$ $\quad \bar{y} = 7.3$

$a_1 = \dfrac{10 \cdot 906 - (73)(105)}{10 \cdot 1477 - 105^2}$, $a_1 = 0.372496622$

$a_0 = 7.3 + a_1 10.5$ $\quad a_0 = 3.388785047$

- Example 2:

x	1	2	3	4	5	6	7
y	0.5	2.5	2.0	4.0	3.5	6.0	5.5

$n = 7$

$\sum x_i = 28$ $\quad \sum x_i y_i = 119.5$ $\quad \bar{y} = 3.428571$

$\sum y_i = 24$ $\quad \sum x_i^2 = 140$ $\quad \bar{x} = 4$

$a_1 = \dfrac{n * \sum x_i y_i - \sum x_i \sum y_i}{n * \sum x_i^2 - (\sum x_i)^2}$

$a_1 = \dfrac{7(119.5) - (28)(24)}{7(140) - 28^2}$, $a_1 = 0.83929$

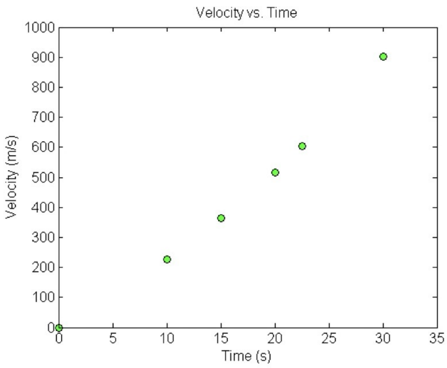
$a_0 = 0.07141$

$y = 0.07141 + 0.83929X$

Error

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using the Lagrangian method for quadratic interpolation.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



$t_1 = 10$ $\quad V_1 = 227.04$ $\quad P_2(t) = l_0(t)\,V_0(t) + \ldots\, l_2(t)\,V_2(t)$

$t_2 = 15$ $\quad V_2 = 362.78$

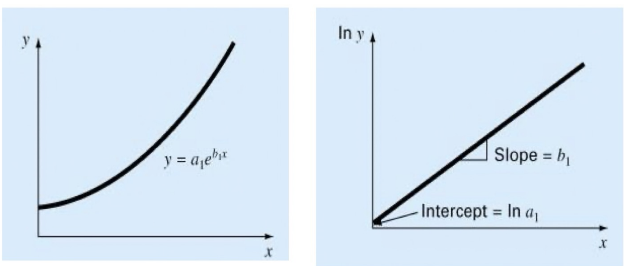
$t_3 = 20$ $\quad V_3 = 517.35$

$$l_1 = \frac{t-t_2}{t_1-t_2}\;\frac{t-t_3}{t_1-t_3} = \frac{16-15}{-5}\;\frac{16-20}{-10} = \cancel{-\frac{1}{5} + \frac{2}{5}} = \frac{-2}{25}$$

$$l_2 = \frac{t-t_1}{t_2-t_1}\;\frac{t-t_3}{t_2-t_3} = 16-10$$

$$l_3 = \frac{t-t_1}{t_3-t_1}\;\frac{t-t_2}{t_3-t_2}$$

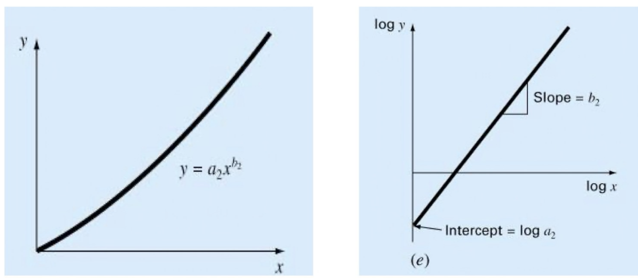
1. The exponential equation.



$$\ln y = \ln a_1 + b_1 x$$

$$y^* = a_o + a_1\, x$$

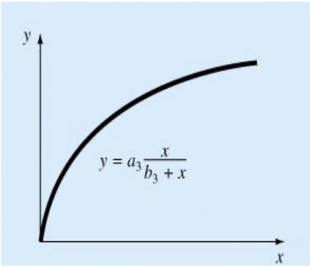
2. The power equation

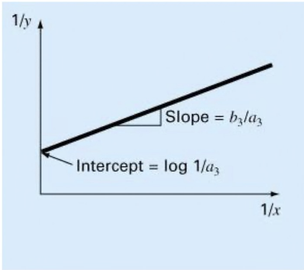


$$\log y = \log a_2 + b_2 \log x$$

$$y^* = a_o + a_1\, x^*$$

3. The saturation-growth-rate equation

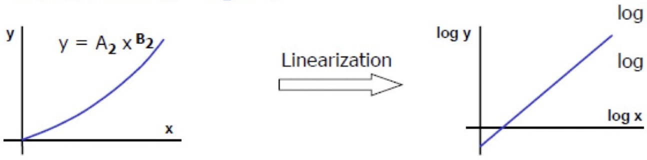




$$\frac{1}{y} = \frac{1}{a_3} + \frac{b_3}{a_3}\left(\frac{1}{x}\right)$$

$y^* = 1/y$
$a_o = 1/a_3$
$a_1 = b_3/a_3$
$x^* = 1/x$

(2) Power Equation ($y = A_2 x^{B_2}$)



$\log y = \log A_2 + B_2 \log x$
or
$\log y = a_0 + a_1 \log x$

Example 4: Fit the following Equation: $y = a_2 x^{b_2}$ to the following data:

$y = a_2 x^{b_2}$, $\log y = \log a_2 + \log x^{b_2}$

$\log y = \log a_2 + b_2 \log x$

let, $y^* = \log y$
$x^* = \log x$
$a_0 = \log a_2$
$a_1 = b_2$

$$\left.\begin{array}{l} y^* = \log y \\ x^* = \log x \\ a_0 = \log a_2 \\ a_1 = b_2 \end{array}\right\} y^* = a_0 + a_1 x^*$$

$y = a_0 - a_1 x$

x_i	y_i
1	0.5
2	1.7
3	3.4
4	5.7
5	8.4
15	19.7

$y(6) = ?$

	X_i	Y_i	$X_i^* = \log(x)$	$Y_i^* = \log(y)$	$X^* Y^*$	X^{*2}
	1	0.5	0	-0.3010	0	0
	2	1.7	0.3010	0.2304	0.0694	0.0906
	3	3.4	0.4771	0.5315	0.2536	0.2276
	4	5.7	0.6021	0.7559	0.4551	0.3625
	5	8.4	0.6990	0.9243	0.6461	0.4886
Sum	15	19.7	2.0792	2.1411	1.4242	1.1693

$$a_1 = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{5 \times (1.4242) - (2.0792)(2.1411)}{5 \times 1.1693 - 4.3231} = 1.7521$$

$a_0 = \bar{y} - a_1 \bar{x}$ $\qquad a_0 = -0.3004$ $\qquad \log y = -0.3004 + 1.7521 \log x$

$$\log y = -0.3004 + \log x^{1.7521}$$

$$\cancel{10^{\log_{10} y}} = 10^{-0.3004} \cdot \cancel{10^{\log_{10} x}}^{1.7621}$$

$$y = 0.43 \cdot x^{1.7521}$$

Euler

Solution:

$f(x,y) = -3xy - y^2 \qquad y(1) = 0.5 \qquad h = 0.1$

$y(1.1) = 0.325$

$$y(x_i + h) = y(x_i) + y'(x_i) \cdot h$$

$$y(1.1) = 0.5 + 0.1(-1.75) = 0.325$$

$$y(1.2) = 0.325 + 0.1(-1.178) = 0.2072$$

2nd Runge-Kutta

$a_1 + a_{21}$

Heun

$$y(x_i + h) = y(x_i) + \left(\tfrac{1}{2} k_1 + \tfrac{1}{2} k_2\right) h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h,\ y_i + k_1 \cdot h)$$

Midpoint

$$y = (x_i + h) = y(x_i) + k_2 \cdot h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{h}{2},\ y_i + \frac{k_1 \cdot h}{2}\right)$$

2^{nd} Runge-Kutta Method: Example

Solve the following system to find $x(1.02)$ using RK2

$\dot{x}(t) = 1 + x^2 + t^3, \qquad x(1) = -4,\ h = 0.01,\ \alpha = 1$

$f(1.01) = y(x_i) + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \qquad f(1.01) = -4 + (9 + 9.0873)$

$k_1 = 1+1+16 = 18 \qquad f(1.01) = -3.819$

$k_2 = f(x_i + h,\ y_i + k_1 . h)$

$k_2 = f(1.01,\ 4 + 0.018) = f(1.01,\ 4.018)$

$k_2 = 18.1746$

$f(1.02) = f(x_i) + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \qquad f(1.01) = -3.815$

$k_1 = f(x_i, y_i) \qquad k_1 = 16.615$

$k_2 = f(x_i + h,\ y_i + k_1 . h) \qquad f(1.02,\ \underbrace{-3.819 + 16.615(0.01)}_{-3.6528}) \qquad k_2 = 15.40416$

$f(1.02) = -3.819 + \left(\frac{16.615}{2} + \frac{15.40416}{2}\right)0.01$

3rd Runge - Kutta

$y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2},\ y_i + \frac{1}{2}k_1 h)$

$k_3 = f(x_i + h,\ y_i + h(k_2 - k_1))$

4th Runge - Kutta

$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_3)$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2},\ y_i + \frac{1}{2}k_1 h)$

$k_3 = f(x_i + \frac{h}{2},\ y_i + \frac{1}{2}k_2 h)$

$k_4 = f(x_i + h,\ y_i + k_3 h)$

Example: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$y' = 1 + y + x^2$

$h = 0.2$

$k_1 = f(x_i, y_i)$, $k_2 = f(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2})$, $k_3 = f(x_i + \frac{h}{3}, y_i + \frac{hk_3}{2})$, $k_4 = f(x_i + h, y_i + k_3 h)$

Solve the differential equation to determine y(1.3) using:

$$\dot{y}(x) + 3xy + y^2 = 0 \qquad y(1) = 0.5 \quad h = 0.1$$

a. Euler Method
b. Second order Runge Kutta method
c. Fourth order Runge-Kutta method
d. Midpoint method

a) Euler: $y_{i+1} = y_i + y'_i \cdot h$

$y(1.1) = y(1) + y'(1)(0.1)$, $y(1.1) = 0.5 - 0.175$ $\quad y(1.1) = 0.325$

$y(1.2) = 0.2072$

$y(1.3) = 0.1283$

b) RK2: $y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$, $k_1 = f(x_i, y_i)$; $k_2 = f(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2})$

$k_1 = -1.75$

$k_2 = f(1.05, 0.4125) = -1.4695$

$y(1.1) = 0.5 + 0.05($

Solve the differential equation to determine y(1.3) using:

$$\dot{y}(x) + 3xy + y^2 = 0 \qquad y(1) = 0.5 \quad h = 0.1$$

a. Euler Method
b. Second order Runge Kutta method
c. Fourth order Runge-Kutta method
d. Midpoint method

d: Midpoint:

$f_{i+1} = f_i + k_2 \cdot h$ $\qquad f_{i+1} = 0.5 + 1.4695 \cdot 0.1$

$k_1 = f(x_i, y) = -1.75$ $\qquad 0.64695$

$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h \cdot k_1}{2})$

$k_2 = f(1.05, \underbrace{0.5 + \frac{0.05 \cdot -1.75}{}}_{0.4125})$

$k_2 = 1.4695$

Solve the differential equation to determine y(1.3) using:

$$y'(x) + 3xy + y^2 = 0 \qquad y(1) = 0.5 \quad h = 0.1$$

$y(1.1) = 0.375$

a. Euler Method

Euler

$y_{i+1} = y_i + y'_i \cdot h$

$y(1.1) = 0.5 + 0.1(-1.75)$

$y(1.1) = 0.5 - 0.125 = 0.375$

$y(1.2) = 0.375 + 0.1($

RK 2

Heun

$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$

$k_1 = y(x_i, y_i)$

$k_2 = y(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2})$

Midpoint

RK4

$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = y(x_i, y_i)$

$k_2 = y(x_i + \frac{h}{4}$