\# Math 207 Midterm I Answer key \#

1) $xy\dfrac{dy}{dx} = 2x^2 + y^2$ --- (Homogeneous differential eqn.

$$\frac{dy}{dx} = \frac{2x^2+y^2}{xy} = \frac{\cancel{x^2}\left(2+\frac{y^2}{x^2}\right)}{\cancel{x^2}\left(\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2+\frac{y^2}{x^2}}{\frac{y}{x}}$$

Let $\frac{y}{x} = v \Rightarrow y = v.x \Rightarrow \frac{dy}{dx} = x.\frac{dv}{dx} + v$

$$\Rightarrow x.\frac{dv}{dx} + v = \frac{2+v^2}{v} \Rightarrow x.\frac{dv}{dx} = \frac{2+v^2-v^2}{v}$$

$$\Rightarrow x.\frac{dv}{dx} = \frac{2}{v} \rightsquigarrow$$ seperable differential eqn.

$$\Rightarrow \int v\,dv = \int \frac{2}{x}\,dx$$

$$\Rightarrow \frac{v^2}{2} = 2\ln x + C,$$ C: any constant.

$$\Rightarrow \boxed{\frac{y^2}{2x^2} = 2\ln x + C}$$ is peneral implicit solution.

2) $\left(\frac{y}{x}+e^{y}-\cos x\right)dx+\left(\ln x+xe^{y}\right)dy=0$ ... (Exact dif...

$M(x,y)=\frac{y}{x}+e^{y}-\cos x \Rightarrow \frac{\partial M}{\partial y}=\frac{1}{x}+e^{y}$

$N(x,y)=\ln x+xe^{y} \Rightarrow \frac{\partial N}{\partial x}=\frac{1}{x}+e^{y}$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ exact differential equation.

$\frac{\partial F}{\partial x}=M(x,y)=\frac{y}{x}+e^{y}-\cos x$

$\Rightarrow F(x,y)=\int\left(\frac{y}{x}+e^{y}-\cos x\right)dx+g(y)$

$F(x,y)=y\ln x+x.e^{y}-\sin x+g(y)$

$\frac{\partial F}{\partial y}=\ln x+xe^{y}+g'(y)=N(x,y)=\ln x+xe^{y}$

$\Rightarrow g'(y)=0 \Rightarrow g(y)=C_1$, C_1 any constant.

$F(x,y)=y\ln x+xe^{y}-\sin x+C_1$

$\Rightarrow C=y\ln x+xe^{y}-\sin x$ is peneral implicit solution

) $x \frac{dy}{dx} + 4y = x^4 y^2$,

$\Rightarrow \frac{dy}{dx} + \frac{4}{x} y = x^3 y^2 \rightarrow$ Bernoulli differential equation with $n=2$

$\Rightarrow$ Multiply both sides by y^{-2}

$y^{-2} \frac{dy}{dx} + \frac{4}{x} y^{-1} = x^3$

Let $v = y^{-1} \Rightarrow \frac{dv}{dx} = - \cdot y^{-2} \frac{dy}{dx} \Rightarrow y^{-2} \frac{dy}{dx} = -\frac{dv}{dx}$

$\Rightarrow -\frac{dv}{dx} + \frac{4}{x} \cdot v = x^3 \Rightarrow \frac{dv}{dx} - \frac{4}{x} v = -x^3 \rightarrow$ Linear diff. eqn.

$p(x) = -\frac{4}{x} \Rightarrow \mu(x) = e^{-4\int \frac{dx}{x}} = e^{-4 \ln x} = x^{-4}$

Integrating factor is x^{-4}

Multiply by the integrating factor gives

$x^{-4} \frac{dv}{dx} - 4x^{-5} v = -x^{-1}$

$\frac{d}{dx} [v \cdot x^{-4}] = -x^{-1}$

$v \cdot x^{-4} = -\int \frac{dx}{x} + c$

$v = (-\ln x + c) \cdot x^4 \Rightarrow v = x^4 \ln \frac{1}{x} + c \cdot x^4$

$\Rightarrow v = y^{-1} \Rightarrow \frac{1}{y} = x^4 \ln x^{-1} + cx^4 \Rightarrow \boxed{y = \frac{1}{x^4 \ln x^{-1} + cx^4}}$

4) a- $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$

$M(x,y) = 3x^2y + 2xy + y^3 \Rightarrow \frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$

$N(x,y) = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact differential equation

$$\mu(x) = e^{\int \frac{\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]}{N(x,y)} dx} = e^{\int \frac{3x^2+2x+3y^2-2x}{x^2+y^2} dx}$$

$$= e^{\int \frac{3x^2+3y^2}{x^2+y^2} dx}$$

$$= e^{\int 3\,dx} = e^{3x}$$

$\Rightarrow \mu(x) = e^{3x}$

b) ⊛ Let y be the amount of radioactive material at any time t. be the amount of radioactive decay.
Then dy/dt represent the rate of radioactive decay. Since decay is proportional to its present amount and the amount material is decreasing

$\frac{dy}{dt} = -ky$ with $y(0) = y_0$ initial amount

$y(2000) = \frac{y_0}{2} \Rightarrow y(4000) = ?$

seperable d.eqn.

$\int \frac{dy}{y} = \int -k\,dt \Rightarrow \ln y = -kt + C_1$

$y = e^{-kt + C_1} \Rightarrow \boxed{y = C.e^{-kt}}$

For $t = 0 \Rightarrow y = y_0 \Rightarrow \boxed{y_0 = C} \Rightarrow \boxed{y = y_0 e^{-kt}}$

$y(2000) = y_0 . e^{-k.2000} = \frac{y_0}{2} \Rightarrow e^{-k} = \left(\frac{1}{2}\right)^{1/2000} \Rightarrow k = \frac{\ln 2}{2000}$

$y(t) = y_0\, e^{-\frac{\ln 2}{2000} . t} \Rightarrow t = 4000 \Rightarrow y(4000) = y_0 . e^{-\frac{\ln 2}{2000} . 4000}$

$= y_0/4.$