24.12.2022

Midterm-2

1. Solve the following IVP by Laplace Transform:

$$y'' + 2y' = \delta(t-1);\ y(0) = y'(0) = 1$$

▸

$$s^2\bar{y}(s) - sy(0) - y'(0) + 2\left[s\bar{y}(s) - y(0)\right] = e^{-s}$$

$$s^2\bar{y}(s) - s - 1 + 2s\bar{y}(s) - 2 = e^{-s}$$

$$s(s+2)\bar{y}(s) = s + 3 + e^{-s} \rightarrow \bar{y}(s) = \frac{s+3}{s(s+2)} + \frac{e^{-s}}{s(s+2)}$$

$$\bar{y}(s) = \frac{s+3}{s(s+2)} + \frac{e^{-s}}{s(s+2)}$$

$$\frac{s+3}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \rightarrow A = \frac{3}{2},\ B = -\frac{1}{2}$$

$$\frac{1}{s(s+2)} = \frac{C}{s} + \frac{D}{s+2} \rightarrow C = \frac{1}{2},\ D = -\frac{1}{2}$$

$$\bar{y}(s) = \frac{3}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{2}\frac{e^{-s}}{s} - \frac{1}{2}\frac{e^{-s}}{s+2}$$

$$y(t) = \frac{3}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}u(t-1) - \frac{1}{2}u(t-1)e^{-2(t-1)}$$

2. Solve the following system of linear first-order IVPs using the eigenvalue/eigenvector approach:

$$x' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} x;\ x(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

▸

$$\det(A - rI) = 0 \rightarrow r_1 = r_2 = r = 1 \rightarrow \beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

$$(A - rI)\gamma = \beta \rightarrow \begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\gamma_1 - 2\gamma_2 = 1$$

$$let \rightarrow \gamma_2 = 1 \rightarrow \gamma_1 = 1 + 2 = 3 \rightarrow x^{(2)} = \beta te^{rt} + \gamma e^{rt} \rightarrow x^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t$$

$$x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t \right\} = \left\{ c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} e^t + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t$$

$$x_1 = (2c_1 + 3c_2)e^t + 2c_2te^t$$

$$x_2 = (c_1 + c_2)e^t + c_2te^t$$

$$\left.\begin{array}{l} x_1(0) = 3 = 2c_1 + 3c_2 \\ x_2(0) = -2 = c_1 + c_2 \end{array}\right\} \rightarrow c_1 = -9,\ c_2 = 7$$

$$x_1 = 3e^t + 14te^t$$

$$x_2 = -2e^t + 7te^t$$

3. Solve the following integro-differential equation:

$$y'(t) + \int_0^t e^{-2\tau} y(t-\tau)d\tau = 3;\; y(0) = 0$$

▸

$$s\bar{y}(s) - y(0) + \bar{y}(s)\frac{1}{s+2} = \frac{3}{s}$$

$$\bar{y}(s)\left(s + \frac{1}{s+2}\right) = \frac{3}{s} \rightarrow \bar{y}(s)\left(\frac{s^2+2s+1}{s+2}\right) = \frac{3}{s}$$

$$\bar{y}(s) = \frac{3s+6}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \rightarrow A = 6, B = -6, C = -3$$

$$y(t) = 6 - 6e^{-t} - 3te^{-t}$$

4. Solve the following first order linear IVP system by *Laplace Transform*:

$$\frac{dx_1}{dt} = -2x_1 + x_2;\quad x_1(0) = 1$$

$$\frac{dx_2}{dt} = 3x_1;\quad x_2(0) = 0$$

▸

$$s\bar{x}_1(s) - 1 = -2\bar{x}_1(s) + \bar{x}_2(s)$$

$$s\bar{x}_2(s) = 3\bar{x}_1(s)$$

$$\left.\begin{array}{l}(s+2)\bar{x}_1(s) - \bar{x}_2(s) = 1\\ -3\bar{x}_1(s) + s\bar{x}_2(s) = 0\end{array}\right\} \rightarrow \begin{array}{l}\bar{x}_1(s) = \dfrac{s}{s^2+2s-3}\\ \bar{x}_2(s) = \dfrac{3}{s^2+2s-3}\end{array}$$

$$\bar{x}_1(s) = \frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1} \rightarrow A = \frac{3}{4},\ B = \frac{1}{4}$$

$$\bar{x}_2(s) = \frac{3}{(s+3)(s-1)} = \frac{C}{s+3} + \frac{D}{s-1} \rightarrow C = -\frac{3}{4},\ B = \frac{3}{4}$$

$$x_1(t) = \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

$$x_2(t) = -\frac{3}{4}e^{-3t} + \frac{3}{4}e^{t}$$

5. Consider the following first order dynamics:

$$\frac{dy}{dt} = t^2 + y; \quad y(1) = 3; \quad 1 \le t \le 1.4$$

Solve the model by *Explicit Euler's Method* and by *Improved Euler's Method* for 2 steps and compare the results by computing the %relative errors using the exact solution $y(t) = 8e^{(t-1)} - 2t - t^2 - 2.$ Use $h = 0.2.$

▸

t	yexp	yimp	yexact	%RE_Explicit	%RE_Implicit
1.0	3	3	3	0	0
1.2	3.8	3.924	3.9312	3.3379	0.18371
1.4	4.848	5.1561	5.1746	6.3116	0.35786

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