13.11.2024

Midterm Exam / Solutions...

1. Find the solution of the following first-order ordinary differential equation:

$$x\frac{dy}{dx} = xe^{-\frac{y}{x}} + y$$

▶

$$y = ux \to y' = u'x + u$$

$$x(u'x + u) = xe^{-u} + ux \to u'x = e^{-u}$$

$$\frac{du}{dx}x = e^{-u} \to \int e^{u}du = \int \frac{dx}{x} \to e^{u} = \ln x - \ln c$$

$$e^{u} = \ln\frac{x}{c} \to \ln(e^{u}) = \ln\left(\ln\frac{x}{c}\right)$$

$$u = \ln\left(\ln\frac{x}{c}\right) \to \frac{y}{x} = \ln\left(\ln\frac{x}{c}\right) \to \boxed{y(x) = x\ln\left(\ln\frac{x}{c}\right)}$$

2. Solve the following initial-value problem:

$$\frac{dy}{dt} - 2ty = ty^2$$

▶

$$\frac{dy}{dt} - 2ty = ty^2 \leftarrow \textit{Bernoulli}$$

$$n = 2 \to u = y^{1-2} = y^{-1} \to \frac{du}{dt} = -y^{-2}\frac{dy}{dt} \to \frac{dy}{dt} = -y^2\frac{du}{dt}$$

$$-y^2\frac{du}{dt} - 2ty = ty^2 \leftarrow \textit{dividing each side by} - y^2 :$$

$$\frac{du}{dt} + 2ty^{-1} = -t \to \boxed{\frac{du}{dt} + 2tu = -t} \to \mu = \exp\int 2tdt = \exp\left[t^2\right] = e^{t^2}$$

$$e^{t^2}\frac{du}{dt} + 2te^{t^2}u = -te^{t^2} \to \frac{d}{dt}\left[ue^{t^2}\right] = -te^{t^2}$$

$$\int d\left[ue^{t^2}\right] = -\int te^{t^2}dt \to ue^{t^2} = -\int te^{t^2}dt$$

$$I = \int te^{t^2}dt \to \omega = t^2 \to d\omega = 2tdt \to tdt = \frac{1}{2}d\omega$$

$$I = \frac{1}{2}\int e^{\omega}d\omega = \frac{1}{2}e^{\omega} = \frac{1}{2}e^{t^2} \to ue^{t^2} = -\frac{1}{2}e^{t^2} + c$$

$$u = -\frac{1}{2} + ce^{-t^2} \to y = \frac{1}{u} \to \boxed{y(t) = \frac{1}{-\frac{1}{2} + ce^{-t^2}}}$$

3. Consider the following first-order initial-value problem:

$$\frac{dy}{dt} = \frac{2y}{t+1}; \quad y(0) = 1$$

Solve the ivp by Explicit Euler's and Heun's methods and comparing them using the Corresponding percent relative errors. Use $h = 0.2$. Perform two iterations.
The exact solution is given by $y_{exact}(t) = (t+1)^2$.

▸

Explicit :

$$y_{n+1} = y_n + \left[\frac{2y_n}{t_n + 1}\right]h$$

$$y_1 = y_0 + \left[\frac{2y_0}{t_0 + 1}\right]h = 1 + \left[\frac{2(1)}{0+1}\right](0.2) = 1.4 \leftarrow t_1 = t_0 + h = 0 + 0.2 = 0.2$$

$$y_2 = y_1 + \left[\frac{2y_1}{t_1 + 1}\right]h = 1.4 + \left[\frac{2(1.4)}{0.2+1}\right](0.2) = 1.8666 \leftarrow t_2 = t_1 + h = 0.2 + 0.2 = 0.2$$

$$Heun: \quad f(t_n, y_n) = \left[\frac{2y_n}{t_n + 1}\right]$$

$$k_1 = \left[\frac{2y_0}{t_0 + 1}\right] = \left[\frac{2(1)}{0+1}\right] = 2$$

$$k_2 = \left[\frac{2y_1}{t_1 + 1}\right] \cong \left[\frac{2(y_0 + k_1 h)}{t_1 + 1}\right] = \left[\frac{2(1 + 2(0.2))}{0.2 + 1}\right] = 2.3333$$

$$y_1 = y_0 + \frac{k_1 + k_2}{2}h = 1 + \frac{2 + 2.3333}{2}(0.2) = 1.4333$$

$$k_1 = \left[\frac{2y_1}{t_1 + 1}\right] = \left[\frac{2(1.4333)}{0.2+1}\right] = 2.3888$$

$$k_2 = \left[\frac{2y_2}{t_2 + 1}\right] \cong \left[\frac{2(y_1 + k_1 h)}{t_2 + 1}\right] = \left[\frac{2(1.4333 + 2.3888(0.2))}{0.4 + 1}\right] = 2.73001$$

$$y_2 = y_1 + \frac{k_1 + k_2}{2}h = 1.4333 + \frac{2.3888 + 2.73001}{2}(0.2) = 1.94518$$

$$y_{exact}(t_1 = 0.2) = 1.44 \rightarrow \begin{cases} \%RE(\exp licit) = \dfrac{|1.44 - 1.4|}{1.44}100 = 2.7777 \\ \%RE(Heun) = \dfrac{|1.44 - 1.433|}{|1.44|}100 = 0.46528 \end{cases}$$

$$y_{exact}(t_2 = 0.4) = 1.96 \rightarrow \begin{cases} \%RE(\exp licit) = \dfrac{|1.96 - 1.8666|}{|1.96|}100 = 4.7653 \\ \%RE(Heun) = \dfrac{|1.96 - 1.94518|}{|1.96|}100 = 0.7561 \end{cases}$$

t	yexpl	Heun	yexact	%RE_Explicit	%RE_Heun
0	1	1	1	0	0
0.2	1.4	1.433	1.44	2.7777	0.46528
0.4	1.8666	1. 94518	1.96	4.7653	0.7561

4. The distance of a ball suspended with a spring from its rest position is given by $y(t)$. The model describing the motion (vibration) is given by the following mathematical model:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 0$$

The ball initially is at rest: (*Initial condition*-1): y(0)=0 $y(0) = 0$

The motion is started with a velocity of $4\,\frac{\text{cm}}{\text{s}}$: (*Initial condition*-2): $\frac{dy}{dt} = 4$.

Find the motion (position) function $y(t)$. What will the response $y(t)$ be *in the long run*?

▸

$$r^2 + 8r + 25 = 0 \rightarrow r_{1,2} = -4 \pm \sqrt{16-25} = -4 \pm 3i \rightarrow \alpha = 4,\quad \beta = 3$$

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \rightarrow \boxed{y(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)}$$

$$y'(t) = -4e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) + e^{-4t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$IC1 \rightarrow 0 = e^0(c_1 \cos(0) + c_2 \sin(0)) \rightarrow \boxed{0 = c_1}$$

$$IC2 \rightarrow 4 = -4e^0((0)\cos(0) + c_2 \sin(0)) + e^0(-3(0)\sin(0) + 3c_2 \cos(0))$$

$$4 = 3c_2 \rightarrow c_2 = \frac{4}{3}$$

$$\boxed{y(t) = \frac{4}{3}e^{-4t} \sin 3t} \rightarrow \lim_{t\to\infty}\left(\frac{4}{3}e^{-4t} \sin 3t\right) = 0 \rightarrow (\textit{stable})$$

NOTE: Due to a syntax error in the exam sheet the second intial condition was taken as $y'(0) = 0$. So the solution depending on this form is ALSO accepted correct.

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \rightarrow \boxed{y(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)}$$

$$y'(t) = -4e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) + e^{-4t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$IC1 \rightarrow 0 = e^0(c_1 \cos(0) + c_2 \sin(0)) \rightarrow \boxed{0 = c_1}$$

$$IC2 \rightarrow 0 = -4e^0((0)\cos(0) + c_2 \sin(0)) + e^0(-3(0)\sin(0) + 3c_2 \cos(0)) \rightarrow \boxed{0 = c_2}$$

$$\boxed{y(t) = 0} \rightarrow (\textit{always stable or no motion})$$

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