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JOINT PROBABILITY DISTRIBUTIONS

Bir olasılık deneyi veya İki olasılık deneyi $\Longrightarrow$ (Random Variable) 2 adet rastgele değişken tanımlanır. (x ve y gibi) $\Longrightarrow$ X ve Y rastgele değişkenlerinin aynı anda gerçekleşmesinin olasılıklarına **JOINT PROBABILITY DISTRUBUTIONS** denir.

* Bir para 2 kez atılıyor ve
* Bir zar 1 kez atılıyor.

$\Longrightarrow$ X: Paranın tura gelmesinin sayısı $X = \{0,1,2\}$

Y: Zarın 3 gelme sayısı $Y = \{0,1\}$

$\Longrightarrow$ $P(x=0, y=0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{24}$

$P(x=0, y=1)$

$\vdots$

DISCRETE JOINT PROBABILITY DISTRIBUTION

* X ve Y iki rastgele değişken olmak üzere, Bu iki rastgele değişkene ait olasılıkların AYNI ANDA gerçekleşmelerine ait olasılıkların dağılımına discrete joint probability distribution denir.

Ex: Bir zar 2 kez, para 3 kez atılıyor :

X: paraların tura gelmesinin sayısı

Y: zarların üst yüzüne 5 gelmesinin sayısı

Buna göre; discrete joint probability distributions ?

$X: \{0,1,2,3\}$

$Y: \{0,1,2\}$

y \ x	0	1	2	3
0	25/288	75/288		
1				
2				

Bu gösterim discrete joint probability dist. dir.

$$P(\underset{\text{hiç tura}}{x=0}, \underset{\text{hiç 5 gelmezek}}{y=0}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{288}$$

$$P(x=1, y=0) = 3\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6} = \frac{75}{288}$$

TYY, YTY, YYT

Discrete Joint Prob. Dis. Özellikleri

① $\sum_{x=i}^{k}\sum_{y=j}^{m} P(X=x, Y=y) = 1$ → Bütün olasılıkların toplamı 1 olmalıdır.

② $P(X=x, Y=y) \geq 0$ → Bütün olasılık değerleri ≥ 0 olmalıdır.

Ex

y \ x	0	1	2
3	c	2c	3c
4	4c	5c	6c

a) c kaçtır?

b) $P(X=2, y=4) = ?$

c) $P(X\leq 1, y=3) = ?$

d) $P(X+Y \leq 4) = ?$

a) $c+2c+3c+4c+5c+6c=1$

$21c=1 \rightarrow c=1/21$

b) $P(X=2, y=4) = 6/21$

c) $P(X=0, Y=3) + P(X=1, Y=3)$

$1/21 + 2/21 = 3/21$

d) $P(X=0, y=4) + P(X=1, Y=3) +$

$P(X=0, Y=3) = 4/21 + 2/21 + 1/21 = 7/21$

Ex:

$$f(x,y) = \begin{cases} \dfrac{a\cdot(x+y)}{5} & , x=0,1 \text{ ve } y=1,2,3 \\ 0 & , \text{diğer durumlar} \end{cases}$$

a) a kaçtır? → $1/3$

b) $P(X=1, Y=2) = ?$ → $1/5$

c) $P(X=0, Y\leq 2) = ?$

y \ x	0	1
1	$a/5$	$2a/5$
2	$2a/5$	$3a/5$
3	$3a/5$	$4a/5$

$3a=1$

$a=1/3$

$P(X=0, y=1) + P(X=0, Y=2) = 1/5$

Ex

Bir torbada 2 mavi, 3 yeşil, 4 kırmızı top vardır.
Bu torbadan 2 top çekiliyor. X: kırmızı top gelme sayısı
y: yeşil top gelme sayısı
Discrete joint prob dis?

$X = \{0,1,2\}$, $y = \{0,1,2\}$

y \ x	0	1	2
0	2/72	16/72	12/72
1	12/72	24/72	0
2	6/72	0	0

$P(x=0, y=0) = 2/9 \cdot 1/8 = 2/72$

$P(x=1, y=0) =$ km veya mk $= 4/9 \cdot 2/8 + 2/9 \cdot 4/8 = 16/72$

$P(x=2, y=0) = 4/9 \cdot 3/8 = 12/72$

$P(x=0, y=1) =$ my veya ym $= 2/9 \cdot 3/8 + 3/9 \cdot 2/8 = 12/72$

$P(x=1, y=1) =$ ky veya yk $= 4/9 \cdot 3/8 + 3/9 \cdot 4/8 = 24/72$

$P(x=2, y=1) = 0$

$P(x=0, y=2) = 3/9 \cdot 2/8 = 6/72$

CONTINOUS JOINT PROBABILITY DISTRIBUTIONS

X ve Y gibi iki farklı rastgele değişkene ait olasılıkların aynı anda gerçekleşmesine ait olasılıkların dağılım fonksiyonudur.

$$f(x,y) = \begin{cases} \text{------} , & 1 \leq x \leq 3 , \ 2 \leq y \leq 5 \\ 0 , & \text{diğer durumlar.} \end{cases}$$

Contionus Joint Probability Distributions Özellikleri

① $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ olmalı

② $f(x,y) \geq 0$ olmalıdır.

$\int^{b}\int^{d} f(x,y)\,dy\,dx$

$$* \; P(a \le X \le b \text{ and } c \le Y \le d) = \int_a \int_c f(x,y)\,dy\,dx$$

Ex

$$f(x,y) = \begin{cases} \overbrace{\tfrac{2}{3}(x+2y)}^{\frac{2x}{3}+\frac{4y}{3}}, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{diğer} \end{cases}$$

fonksiyonunun cont. joint prob. dis. olup olmadığını bulunuz.

1. şart

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \qquad \int_0^1\int_0^1 \tfrac{2}{3}(x+2y)\,dx\,dy = 1 \checkmark$$

2. şart

$f(x,y) \geq 0$

$$\tfrac{2}{3}/\ 0 \le x \le 1 \rightarrow 0 \le \frac{2x}{3} \le \frac{2}{3}$$

$$\tfrac{4}{3}/\ 0 \le y \le 1 \rightarrow 0 \le \frac{4y}{3} \le \frac{4}{3}$$

$$+ \quad 0 \le \underbrace{\frac{2x}{3} + \frac{4y}{3}}_{} \le 2 \rightarrow f(x,y) \geq 0 \checkmark$$

Ex

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \rightarrow \int_0^1\int_0^1 \frac{4x}{5} + \frac{6y}{5}\,dx\,dy = \frac{2x^2}{5} + \frac{6xy}{5}\Big|_0^1 = \frac{2}{5} + \frac{6y}{5}$

2-11

$\int_0^1 \frac{2}{5} + \frac{6y}{5}\,dy = \frac{2y}{5} + \frac{3y^2}{5}\Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1 \checkmark$

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

✓(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

b) $\int_0^{1/2}\int_{1/4}^{1/2} \frac{4x}{5} + \frac{6y}{5}\,dy\,dx$

NOT: $P(X=2, Y=3) = 0$
$P(X=2, Y\le 3) = 0$ } Cont. joint prob. dis. de eşitlik varsa cevap hep 0 olur!

Eşitlik yoksa cevap 0'dan farklı çıkar.

Ex

$$f(x,y) = \begin{cases} \frac{1}{6}(x+y), & 1 \le x \le 2,\ 4 \le y \le 5 \\ 0, & \text{diğer} \end{cases}$$

a) $P(X=1{,}5,\ Y=4{,}2) = ? \rightarrow 0$

b) $P(X < 3/2,\ Y \le 9/2) = ? \rightarrow \displaystyle\int_1^{3/2}\int_4^{9/2} \left(\frac{x}{6}+\frac{y}{6}\right) dy\,dx$

c) $P(-2 < X < 1{,}2,\ Y=4) = ? \rightarrow 0$

d) $P(1 < X,\ Y < 9/2) = ?$

$$\hookrightarrow \int_1^{2}\int_4^{9/2} \frac{x}{6}+\frac{y}{6}\, dy\,dx$$

MARGINAL PROBABILITY DISTRIBUTIONS

→ DISCRETE MARGINAL

Ex

$y \backslash x$	0	1	2	
1	1/5	1/10	0	3/10
2	3/10	0	1/10	4/10
3	1/10	1/10	1/10	3/10
	$\frac{6}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	

X ve Y'nin marginal olasılık dağılımlarını bulunuz

X	0	1	2
P(X=x)	$\frac{6}{10}$	$\frac{2}{10}$	$\frac{2}{10}$

Y	1	2	3
P(Y=y)	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

→ CONTINOUS MARGINAL

* X'in marginal Dağılım Fonk =

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy}$$

* Y'nin marginal Dağılım Fonk. =

$$f(y) = \int^{\infty} f(x,y)\,dx$$

$-\infty$

Ex

$$f(x,y) = \begin{cases} \frac{1}{6}(x+y) \, , & 1 \le x \le 2 \, , \, 4 \le y \le 5 \\ 0 \, , & \text{diğer} \end{cases}$$

a) X'in marginal dağılım fonk?

b) y'nin marginal dağılım fonk?

a) $f(x) = \int_4^5 \frac{x}{6} + \frac{y}{6} \, dy$

$\downarrow$

$$f(x) = \begin{cases} \frac{x}{6} + \frac{3}{4} \, , & 1 \le x \le 2 \\ 0 \, , & \text{diğer} \end{cases}$$

b) $f(y) = \int_1^2 \frac{x}{6} + \frac{y}{6} \, dx$

$\downarrow$

$$f(y) = \begin{cases} \frac{1}{4} + \frac{y}{6} \, , & 4 \le y \le 5 \\ 0 \, , & \text{diğer} \end{cases}$$

Ex

A production facility contains two machines that are used to rework items that are initially defective. Let X be the number of hours that the first machine is in use, and let Y be the number of hours that the second machine is in use, on a randomly chosen day. Assume that X and Y have joint probability density function given by

$$f(x) = \begin{cases} \frac{3}{2}(x^2 + y^2) & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the probability that both machines are in operation for more than half an hour?

b. Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

a) $P(x > 0{,}5 \, , \, y > 0{,}5) = \int^1 \int^1 \frac{3}{2}(x^2+y^2) \, dx \, dy$

0,5 0,5

b) $f(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dy \longrightarrow f(x) = \begin{cases} \text{-----}, & 0<x<1 \\ 0, & \text{diğer} \end{cases}$

$f(y) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dy \longrightarrow f(y) = \begin{cases} \text{-----}, & 0<y<1 \\ 0, & \text{diğer} \end{cases}$

* $\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

* $P(a<X<b) = P(a<X<b, -\infty<Y<\infty)$

$$= \int_a^b \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_a^b g(x)\,dx$$

CONDITIONAL DISTRIBUTIONS

* Y given that $X=x$:

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

* X given that $Y=y$:

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

b

$$* \; P(a<X<b \mid Y=y) = \int_a f(x|y)\,dx$$

The conditional probability mass function of Y given $X=x$ is (discrete)

$$P_{Y|X}(y|x) = \frac{P(x,y)}{P(X)}$$

Ex

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find the conditional distribution of X, given that $Y=1$ and use it to determine $P(X=0|Y=1)$.

	Y=0	Y=1	Y=2	Row Total
X=0	3/28	6/28	1/28	10/28
X=1	9/28	6/28	0	15/28
X=2	3/28	0	0	3/28
Column Total	15/28	12/28	1/28	1

a) $P_{X|Y}(x|y=1) = ?$

a) $\cdot P_{X|Y}(X=0|y=1) = \dfrac{P(x=0, y=1)}{P(y=1)} = \dfrac{6/28}{12/28} = 1/2$

$\cdot P_{X|Y}(X=1|y=1) = \dfrac{P(x=1, y=1)}{P(y=1)} = \dfrac{6/28}{12/28} = 1/2$

$\cdot P_{X|Y}(x=2|y=1) = \dfrac{P(x=2, y=1)}{P(y=1)} = 0$

a da isteneni bulmak için bunları ayrı ayrı bulmalıyız.

b) $P_{Y|X}(y|x=0) = ?$

b) $\cdot P_{Y|X}(y=0|x=0) = \dfrac{P(y=0, x=0)}{P(x=0)} = \dfrac{3/28}{10/28} = 3/10$

$\cdot P_{Y|X}(y=1|x=0) = P(y=1, x=0) \quad 6/28 \quad 6/$

b'de isteneni bulmak için

$\cdot\ P_{Y|X}(y=1 \mid x=0) = \dfrac{P(y=1, x=0)}{P(x=0)} = \dfrac{9/28}{10/28} = 9/10$

$\cdot\ P_{Y|X}(y=2 \mid x=0) = \dfrac{P(y=2, x=0)}{P(x=0)} = \dfrac{1/28}{10/28} = 1/10$

bu olasılıkları ayrı ayrı bulduk.

The conditional probability density function of Y given X = x
(continous)

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)}$$

Ex

(Continuing **Example 2.54**.) The joint probability density function of the thickness X and hole diameter Y (both in millimeters) of a randomly chosen washer is $f(x, y) = (1/6)(x + y)$ for $1 \leq x \leq 2$ and $4 \leq y \leq 5$. Find the conditional probability density function of Y given $X = 1.2$. Find the probability that the hole diameter is less than or equal to 4.8 mm given that the thickness is 1.2 mm.

a) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f(x)}$ $\quad f(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$

$$f(x) = \int_4^5 \frac{x}{6} + \frac{y}{6}\,dy = \left.\frac{xy}{6} + \frac{y^2}{12}\right|_4^5 = \frac{1}{6}\left(x + \frac{9}{2}\right)$$

$$f_{Y|X}(y|x) = \frac{\frac{1}{6}(x+y)}{\frac{1}{6}\left(x+\frac{9}{2}\right)} \qquad \int_a^b f(y|x)\,dy$$

$$P(Y=y \mid X=1.2) = \int_4^5 \frac{x+y}{x+\frac{9}{2}}\,dy$$

$x \to 1.2$

b) $$P(Y \leq 4.8,\ X = 1.2) = \int_4^{4.8} \frac{1.2+y}{1.2+\frac{9}{2}}\,dy$$

Ex

The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & elsewhere. \end{cases}$$

a) Find the marginal densities $f_X(x)$, $f_Y(y)$, and the conditional density $f_{X|Y}(x|y)$.

b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

a)

$$f_x(x) = \int_x^1 10xy^2\,dy = \left.\frac{10xy^3}{3}\right|_x^1 = \frac{10x - 10x^4}{3}$$

$$f_y(y) = \int_0^y 10xy^2\,dx = \left.5x^2y^2\right|_0^y = 5y^4$$

$$f_{x|y}(x|y) = \frac{f(x,y)}{f(y)} = \frac{10xy^2}{5y^4}$$

b) $P(y \geq 0.5 \mid x = 0.25) = ?$

$$f_{y|x}(y|x) = \frac{f(x,y)}{f(x)} = \frac{10xy^2}{\frac{10x-10x^4}{3}} = \frac{3y^2}{1-x^3}$$

$$\int_{0.5}^{1} \frac{3y^2}{1-x^3}\,dy = \int_{0.5}^{1} \frac{3y^2}{1-(0{,}25)^3}\,dy = \int_{0.5}^{1} \frac{3y^2}{0{,}98}\,dy$$

STATISTICAL INDEPENDENCE

* $f(x,y) = g(x) \cdot h(y)$ → eğer eşitlik sağlanıyorsa **statically independent** deriz.

Ex

$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$? statically independent mı?

→ $\underbrace{P(x=0, y=0)}_{3/28} = \underbrace{P(x=0) \cdot P(y=0)}_{10/28 \cdot 15/28}$

→ not equal → not independent (Dependent)

→ Eşitlik olmadığı için durduk, olsaydı tüm olasılıkları teker teker denerdik.

MEAN OF RANDOM VARIABLE

→ Let X be a random variable with probability distribution f(x). The **mean**, or **expected value** or **average** of X is;

* If X is discrete;

$$\boxed{\mu = E(x) = \sum_x x \cdot f(x)} \longrightarrow \boxed{\mu_x = \sum_x x \cdot P(X=x)}$$

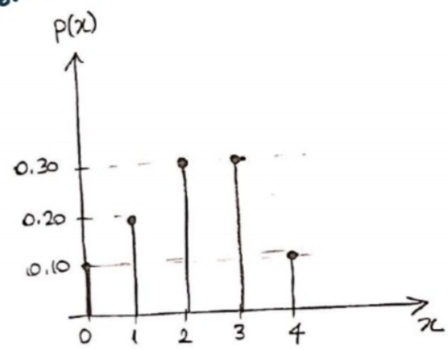
* If X is continous;

$$\boxed{\mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

Interpretation

corresponding probability

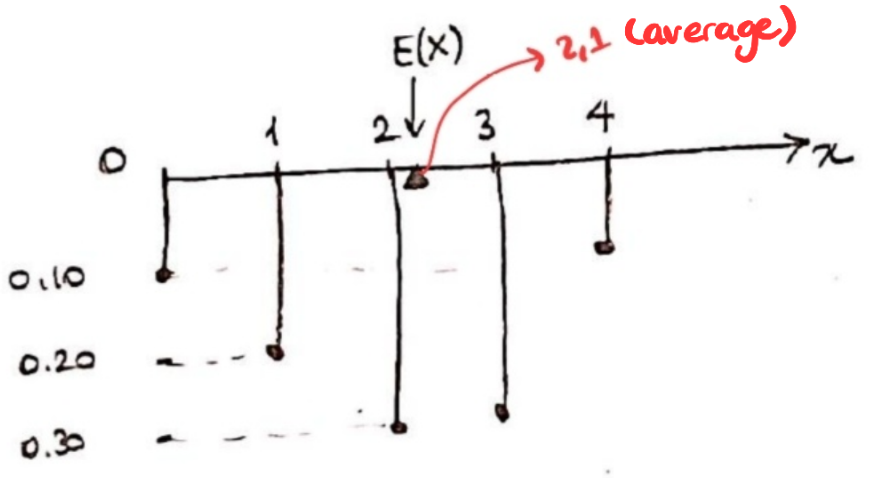
Number of cars sold x	$P(x)$
0	0,10
1	0,20
2	0,30
3	0,30
4	0,10



$E(x) = 0 \cdot (0,10) + 1 \cdot (0,20) + 2 \cdot (0,30) + 3 \cdot (0,30) + 4 \cdot (0,10) = 2,10$

Expected value of X

Center of Gravity



Ex

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

\# of good component

$X = \{0, 1, 2, 3\}$ $\quad E(X) = ?$

$$P(X=3) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

$$* \; P(X=0) = \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$$

$$* \; E(X) = \sum_{x=0}^{3} P(X=x) \cdot x$$

$$P(X=1) = \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$$

$$= 0 \cdot \frac{1}{35} + 1 \cdot \frac{12}{35} + 2 \cdot \frac{18}{35} + 3 \cdot \frac{4}{35} = \underline{\underline{1{,}7}}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$$

McGraw-Hill

Ex

$$f(x) = \begin{cases} \frac{2x}{15}, & 1 < x < 4 \\ 0, & \text{diğer} \end{cases}$$

sürekli olasılık dağılımının expected value'su $E(X) = ?$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$E(X) = \int_{1}^{4} x \cdot \frac{2x}{15}\, dx = \left. \frac{2x^3}{3 \cdot 15} \right|_{1}^{4}$$

Ex

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

↳ we can't cound (continous)

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Find the expected life of this type of device.

$$E(x) = \int_{100}^{\infty} x \cdot \frac{20.000}{x^3}\, dx = \lim_{A \to \infty} \int_{100}^{A} \frac{20.000}{x^2}\, dx = 20.000 \lim_{A \to \infty} (-x^{-1}) \Big|_{100}^{A}$$

$$20.000 \lim_{A \to \infty} \left(\frac{1}{100} - \underbrace{\frac{1}{A}}_{0} \right) = \frac{20.000}{100} = 200 \text{ hour.}$$

→ Means of Functions of Random Variables

* If X is discreate ;

$$\boxed{\mu_{h(x)} = \sum_x h(x) \cdot p(x)}$$

p(x) : probability mass function
h(x) : given

* If X is continous ;

$$\boxed{\mu_{h(x)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx}$$

f(x) : probability density function
h(x) : given

THEOREM

→ If X is discrete ;

$$\boxed{\mu_{g(x)} = E(g(x)) = \sum_x g(x) \cdot f(x)}$$

random variable g(x)

→ If X is continous ;

$$\mu_{g(x)} = E(g(x)) = \int^{\infty} g(x) \cdot f(x)\, dx$$

$$\mu_{g(x)} = E(g(x)) = \int_{-\infty} g(x).f(x)\,dx$$

Ex

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

$4 \to g(x)=7$
$5 \to g(x)=9$
$6 \to g(x)=11$

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

$$E(g(x)) = \sum_x g(x).f(X=x)$$

$$= 7.\frac{1}{12} + 9.\frac{1}{12} + 11.\frac{1}{4} + 13.\frac{1}{4} + 15.\frac{1}{6} + 17.\frac{1}{6} = 12.67 \$$$

Ex

An internal combustion engine contains several cylinders bored into an engine block. Let X denote the bore diameter of a cylinder, in millimeters. Assume that the probability distribution of X is

we can't count = continous

$$f(x) = \begin{cases} 10, & 80.5 < x < 80.6 \\ 0, & \text{otherwise} \end{cases}$$

$E(A(x)) = ?$

Let $A = \pi X^2/4$ represent the area of the bore. Find the mean of A.

$$E(A(x)) = \int_{80.5}^{80.6} \pi \frac{x^2}{4}.10\,dx = \frac{10}{4}\pi.\frac{x^3}{3}\Big|_{80.5}^{80.6}$$

VARIANCE OF RANDOM VARIABLES

→ **For Discrete:** Varince → σ^2, $V(x)$ veya $Var[x]$ gibi gösterilebilir.

$$\boxed{\begin{aligned} \sigma^2 &= E(x^2) - (E(x))^2 \\ \sigma^2 &= E(x^2) - \mu^2 \end{aligned}}$$

$$E(x^2) = \sum_x x^2.f(x)$$

$$E(x) = \sum_x x.f(x)$$

$$\boxed{\sigma^2 = E\left[(x-\mu)^2\right] = \sum_x (x-\mu)^2 \cdot f(x)}$$

$$\sigma_x^2 = \sum_x (x-\mu_x)^2 \cdot P(X=x) = \sum_x x^2 \cdot P(X=x) - \mu_x^2$$

Positive root of variance

$$\boxed{\sigma = \sqrt{\sigma^2}}$$!

Standart deviation of x olarak geçer. !

Ex

x	1	2	3
P(X=x)	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{2}{7}$

X rastgele değişkenine ait kesikli olasılık dağılımı verilmiştir.

Varyans nedir? $\sigma^2 = ?$

$\sigma^2 = E(x^2) - (E(x))^2$

$E(x^2) = 1^2 \cdot \frac{3}{7} + 2^2 \cdot \frac{2}{7} + 3^2 \cdot \frac{2}{7} = \frac{29}{7}$

$E(x) = 1 \cdot \frac{3}{7} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{2}{7} = \frac{13}{7}$

$\sigma^2 = \frac{29}{7} - \left(\frac{13}{7}\right)^2 = \frac{34}{49}$

Ex

A certain industrial process is brought down for recalibration whenever the quality of the items produced falls below specifications. Let X represent the number of times the process is recalibrated during a week and assume that X has the following probability mass function.

x	0	1	2	3	4
$p(x)$	0.35	0.25	0.20	0.15	0.05

Find the mean and variance of X.

$\mu = 0 \cdot (0,35) + 1 \cdot (0,25) + 2 \cdot (0,20) + 3 \cdot (0,15) + 4 \cdot (0,05)$

$\mu = 1,3 = E(x)$

$E(x^2) = 0^2 (0,35) + 1^2 (0,25) + 2^2 (0,20) + 3^2 (0,15) + 4^2 (0,05)$

$E(x^2) = 3,2$

$\sigma^2 = E(x^2) - \mu^2 = 3,2 - (1,3)^2 = 1,51$

$* \ \text{std}x = \sqrt{\sigma^2}$

$= \sqrt{1,51} = 1,23$

Ex

A resistor in a certain circuit is specified to have a resistance in the range 99 Ω–101 Ω. An engineer obtains two resistors. The probability that both of them meet the specification is 0.36, the probability that exactly one of them meets the specification is 0.48, and the probability that neither of them meets the specification is 0.16. Let X represent the number of resistors that meet the specification. Find the probability mass function, and the mean, variance, and standard deviation of X.

we can count # of resistence (discrete)

$P(x=0) = 0{,}16$
$P(x=1) = 0{,}48$
$P(x=2) = 0{,}36$

$\mu = E(x) = 0\cdot(0{,}16) + 1\cdot(0{,}48) + 2\cdot(0{,}36)$
$\mu = 1{,}2$

$E(x^2) = 0^2\cdot(0{,}16) + 1^2(0{,}48) + 2^2(0{,}36)$
$E(x^2) = 1{,}92$

$\sigma^2 = E(x^2) - (E(x))^2 = 1{,}92 - (1{,}2)^2 = 0{,}48$

$std(x) = \sqrt{\sigma_x^2} = \sqrt{0{,}48} = 0{,}69$

→ For Continous:

$$\boxed{\begin{aligned}\sigma^2 &= E(x^2) - (E(x))^2 \\ \sigma^2 &= E(x^2) - \mu^2\end{aligned}} \rightarrow E(x^2) = \int_{-\infty}^{\infty} x^2 . f(x)\, dx \;, \quad \mu = \int_{-\infty}^{\infty} x . f(x)\, dx$$

Ex

$$f(x) = \begin{cases} 2x/15 & , 1 < x < 4 \\ 0 & , \text{diğer} \end{cases}$$

şeklinde verilen sürekli olasılık dağılımının $\sigma^2 = ?$

$\sigma^2 = E(x^2) - (E(x))^2$

$$E(x) = \int_1^4 x . \frac{2x}{15}\, dx = \left.\frac{2x^3}{45}\right|_1^4 = \frac{14}{5}$$

$$E(x^2) = \int_1^4 x^2 . \frac{2x}{15}\, dx = \left.\frac{2x^4}{60}\right|_1^4 = \frac{17}{2}$$

$$\sigma^2 = \frac{17}{2} - \left(\frac{14}{5}\right)^2$$

Ex

A hole is drilled in a sheet-metal component, and then a shaft is inserted through the hole. The shaft clearance is equal to the difference between the radius of the hole and the radius of the shaft. Let the random variable X denote the clearance, in millimeters. The probability density function of X is

$$f(x) = \begin{cases} 1.25(1 - x^4), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean clearance and the variance of the clearance.

$$\mu = E(x) = \int_0^1 x \cdot (1{,}25)(1-x^4)\,dx = 0{,}83$$

$$E(x^2) = \int_0^1 x^2 \cdot (1{,}25)(1-x^4)\,dx = 0{,}71$$

$$Var(x) = \sigma^2 = E(x^2) - (E(x))^2$$
$$= 0.71 - (0{,}83)^2$$
$$= 0{,}024$$

VARIANCE of FUNCTIONS of RANDOM VARIABLES

→ Let X be a random variable with probability distribution f(x). The variance of the random variable g(x) is

* **X is Discrete:**

$$\boxed{\sigma^2 = E[g(x) - \mu_{g(x)}]^2 \qquad \sigma^2 = E[g(x) - E(g(x))]^2}$$

* **X is Continous:**

$$\boxed{\sigma^2 = E[g(x) - \mu_{g(x)}]^2 \qquad \sigma^2 = E[g(x) - E(g(x))]^2}$$

Ex

$$\sigma^2 = E[g(x) - E(g(x))]^2$$

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

$$\mu_{g(x)} = \sum_x g(x) \cdot f(x)$$

$g(0) = 3$
$g(1) = 5$
$g(2) = 7$
$g(3) = 9$

$$\mu_{g(x)} = 3 \cdot \frac{1}{4} + 5 \cdot \frac{1}{8} + 7 \cdot \frac{1}{2} + 9 \cdot \frac{1}{8} = 6$$

$\sigma^2 = E\left(g(x) - E(g(x))\right)^2 = (2x+3-6)^2 = (2x-3)^2 = E(4x^2-12x+9)$

$$\sum_{x}^{3} (4x^2-12x+9).f(x) = 9.\frac{1}{4} + 1.\frac{1}{8} + 1.\frac{1}{2} + 9.\frac{1}{8}$$

Ex

$$= \underset{(2)}{\frac{9}{4}} + \frac{1}{8} + \underset{(4)}{\frac{1}{2}} + \frac{9}{8} = \frac{18+1+4+9}{8} = \frac{32}{8} = 4$$

Let X be a random variable having the density function given in Example 4.5 on page 135. Find the variance of the random variable $g(X) = 4X + 3$.

Let X be a random variable with density function

$$\mu_{g(x)} = \int_{-\infty}^{\infty} g(x).f(x)\,dx \qquad f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$E(g(x))$

$$\int_{-1}^{2} (4x+3).\frac{x^2}{3}\,dx = 6 \qquad \sigma^2 = \int_{-1}^{2} [(4x+3)-6]^2\,dx = 39$$

BINOMIAL and MULTINOMINAL DISTRIBUTIONS

→ Bir olasılık değerinde 2 olası sonuç varsa buna Bernoulli Distribution denir.

* Bir zar atma bernoulli değildir $\{1,2,3,4,5,6\}$ X
* Bir para atma bernoullidir. { Tura, Yazı } ✓
* Bir hastalıktan iyileşme veya iyileşmeme bernoullidir. ✓

→ Bernoulli Distribution:

- There are single trial
- The trial can result in one of two possible outcomes.
- $P(\text{success}) = p$
- $P(\text{failure}) = 1-p$

→ For any Bernouilli trial, if we define random variable X :

- If the experiment result in a success → $X=1$
- Otherwise → $X=0$

* $P(0) = P(X=0) = 1-p$ (probability of failure)
* $P(1) = P(X=1) = p$ (probability of success)
* $P(x) = 0 \rightarrow$ for any value of x other than 0 or 1

$$\boxed{P(X=x) = p^x \cdot (1-p)^{(1-x)} \quad ; x = 0,1}$$

→ For Bernoulli ;

$\mu = p$ (success)

$\sigma^2 = q \cdot p = (1-p) \cdot q$

Ex

Ten percent of components manufactured by a certain process are defective. A component is chosen at random. Let $X = 1$ if the component is defective, and $X = 0$ otherwise.

0,1 → defective
0,9 → non-defective

1. What is the distribution of X?

(non-defective)
$X \sim$ Bernoulli (p) ⇒ $P(X=0)$ = probability of failure → 0,9
$P(X=1)$ = probability of success (defective) → 0,1

2. Find the mean and variance of X.

$E(x) = \mu = p =$ success $= 0,1$
$\sigma^2 = p \cdot q = (0,1)(0,9) = 0,09$

→ THE BINOMIAL DISTRIBUTION

* n defa bernoulli deneyi gerçekleştirilirse BINOMIAL DISTRIBUTION denir.
* The trials are independent
* Each trial has the same success probability p

n defa x başarı

p : başarı olasılığı
q : başarısızlık olasılığı

$\boxed{p+q=1}$

Binom dağılımında → $\binom{n}{x} \cdot p^x \cdot q^{n-x}$ → Bin(n, p) olasılık hesaplama $\binom{}{x}$

Ex

Bir para 7 kez atılıyor. 3 kez yazı gelme olasılığı nedir?

yazı = p = 1/2

tura = q = 1/2

$$\underbrace{\binom{7}{3}}_{\binom{n}{x}} \cdot \underbrace{\left(\frac{1}{2}\right)^3}_{p^x} \cdot \underbrace{\left(\frac{1}{2}\right)^{7-3}}_{q^{n-x}} = 35 \cdot \frac{1}{8} \cdot \frac{1}{16} = \frac{35}{128}$$

Ex

Bir hastanede bir hastalıktan iyileşme olasılığı 0,07'dir. Bu hastaneye bu hastalıktan gelen 10 kişinin

a) 2'sinin iyileşme olasılığı nedir?

b) 6'sının iyileşme olasılığı nedir?

c) En fazla 2'sinin iyileşme olasılığı nedir?

d) En az 3'ünün iyileşme olasılığı nedir?

a) p = 0,07, q = 0,93 $\binom{10}{2} \cdot (0{,}07)^2 (0{,}93)^8$

b) $\binom{10}{6} \cdot (0{,}07)^6 \cdot (0{,}93)^4$

c) 0 kişi, 1 kişi, 2 kişi → $\binom{10}{0} \cdot (0{,}07)^0 (0{,}93)^{10} + \binom{10}{1} (0{,}07)^1 (0{,}93)^9 + \binom{10}{2} (0{,}07)^2 (0{,}93)^8$

d) 3 kişi, 4 kişi, ..., 10 kişi → 1 − istenmeyen

$1 - \binom{10}{0} (0{,}07)^0 (0{,}93)^{10} - \binom{10}{1} \cdot (0{,}07)^1 (0{,}93)^9 - \binom{10}{2} (0{,}07)^2 (0{,}93)^8$

* μ = mean = expected value = $n \cdot p$
  σ^2 = variance = $n \cdot p \cdot q$ $(q = 1-p)$
  Standart derision = $\sqrt{\sigma^2} = \sqrt{n \cdot p \cdot q}$

Ex

Bir hastanede bir hastalıktan iyileşme olasılığı 0,03'tür. Bu hastaneye yatan 100 kişiden iyileşenlerin μ sü ve σ^2 kaçtır?

$\mu = n.p = 100.(0,03) = 3$

$\sigma^2 = n.p.q = 100.(0,03).(0,97) = 2,91$

Ex

Sekiz yüzlü bir zarın yüzlerinde 1, 1, 2, 2, 2, 3, 4 ve 6 bulunmaktadır. Bu zar 100 defa atıldığında üst yüze 2 gelmesinin beklenen değeri kaçtır?

istenen → 2 gelmesi $= 3/8$

istenmeyen → 2 gelmemesi $= 5/8$

$\mu = E(x) = n.p = 100 . 3/8 = 75/2 = 37,5$

$\sigma^2 = n.p.q = 100 . 3/8 . 5/8$

Ex

Bir piyango çekilişinde bir bilete ikramiye çıkma olasılığı $1/10$'dur.

a) İkramiye çıkan bir bilet bulabilmek için ortalama (mean) kaç bilet satın alınmalıdır?

b) Satın alınan 20 bilette 2 tane ikramiye çıkan bilet olma olasılığını bulun.

a) $\mu = n.p = 1 \rightarrow n . 1/10 = 1 \rightarrow n = 10$

b) $\binom{20}{2} . (1/10)^2 . (9/10)^{18}$

Ex

Bir şehirde bulunan evlerin 95%'inde TV bulunmaktadır. Bu şehirden seçilen 20 evden 2 sinde TV bulunmama olasılığı kaçtır?

↳ 18 tanesinde var

$p = 0,95$ (TV olma)

$q = 0,05$ (TV olmama)

$\binom{20}{18} (0,95)^{18} (0,05)^2$

Ex

A fair coin is tossed 10 times. Let X be the number of heads that appear. What is the distribution of X?

X = # number of heads

$n = 10$

p: probability of success = probability of heads = 0.5

$X \sim \text{Binom}(\underset{n}{\underset{\downarrow}{10}}, \underset{p}{\underset{\downarrow}{0.5}})$

INDEPENDENCE of TRIALS

→ When selecting items from a box, you have to replace the item you selected, to make your selections independent. (with replacement)

→ There are some cases where your selections are independent even though the selections are made without replacement:

- Selections from an infinite population (sonsuz)
- Selections from finite population (sonlu)

! Bir finite population olsun → 2 type'ı var (success ve failure)

↳ Bu population'dan bir simple random sample seçiyorsun

↳ Bu seçtiğin sample'ın size'ı population'un size'ının 5 %'inden küçükse;

↳ Yani; sample size < 0,05 (population size)

↳ Burda Binomial Distribution var diyebiliriz.

Ex

2-14

A lot contains several thousand components, 10% of which are defective. Seven components are sampled from the lot. Let X represent the number of defective components in the sample. What is the distribution of X?

- Since the sample size (→ seven components) is small compared to the population (i.e., less than 5%), the number of successes in the sample approximately follows a binomial distribution.
- Therefore we model X with the Bin(7, 0.1) distribution.

→ Binomial distribution

$\underset{n}{7}$, $\underset{p}{0.1}$

P(x=1) → probability of success → probability of defective = 0,1

→ several thousand component için 7 component

Sample size çok küçük < 0,05

↳ Bu nedenle binomial

Ex

2-17

A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%. → probability of defective item = 0,03

a) The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

↳ probability of non-defective item = 0,97

b) Suppose that the retailer receives 10 shipments in a month and the inspector randomly tests 20 devices per shipment. What is the probability that there will be exactly 3 shipments each containing at least one defective device among the 20 that are selected and tested from the shipment?

a) P(probability of at least one defective) = 1 − (probability of all 20 item being non-defective)

$= 1 - (0,97)^{20} = 0,456$

→ $P(X=0) = \binom{20}{0}(0,03)^0(0,97)^{20}$

b) $\left.\begin{matrix} n=10 \\ p=0,456 \end{matrix}\right\}$ $P(X=3) = \binom{10}{3}(0,456)^3(0,544)^7$

Ex

2-16

A large industrial firm allows a discount on any invoice that is paid within 30 days. Of all invoices, 10% receive the discount. In a company audit, 12 invoices are sampled at random.

a) What is the probability that fewer than 4 of the 12 sampled invoices receive the discount?

b) What are the mean and variance of the number of invoices receive the discount?

X = # of invoices receive the discount

p = probobility of success = probability of receive the discount = 10% = 0,1

$X \sim Bin(12, 0.1)$

a) $P(X<4) = ? \rightarrow P(X=3) + P(X=2) + P(X=1) + P(X=0)$

$\binom{12}{3}(0,1)^3(0,9)^9 + \binom{12}{2}(0,1)^2(0,9)^{10} + \binom{12}{1}(0,1)^1(0,9)^{11} + \binom{12}{0}(0,1)^0(0,9)^{12}$

b) $\mu = E(X) = n.p = 12.(0,1)$

$\sigma^2 = n.p.q = 12.(0,1)(0,9)$

→ **MULTINOMIAL TRIALS**

→ n tane Bernoulli deneyimiz olduğunu varsayalım.

↳ Her bir deneyin olasılığı p olsun.

↳ $y_1 \ldots y_n \rightarrow$ $y_i = 1$ (ith trial result is success)
$y_i = 0$ (ith trial result is failure)

↳ X : n denemede oluşan success sayısı olsun.

↳ $X = y_1 + y_2 + \ldots + y_n$

Binomial bir Random Variable'ı Bernoulli'lerin toplamı olarak gösterebiliriz.

→ Bernoulli trial'da 2 tane olası outcome vardı. Ama;

→ Bernoulli'lerin toplamı sonucu oluşan yeni binomial'da 2'den fazla outcome olabilir. $k \geq 2$ (k : binomun outcome sayısı)

↳ k outcomes ⟶ $p_1, \ldots, p_k$

↳ Bernoulli'lerin toplanması olayına → Multinomial Trials denir.

→ Number of outcomes → 1, 2, ..., k

↳ Her bir outcome için i → X_i → i'nin random variable'ı

↳ $X_1, X_2, \ldots, X_k$ → discrete random variable

↳

Bunlara Multinomial Distribution denir.

$$\boxed{X_1, X_2, \ldots, X_k \sim MN(n, p_1, p_2, \ldots, p_k)}$$

!

$$P(x) = P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$$

$$= \frac{n!}{x_1!\, x_2! \ldots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$$

!

Ex

The items produced on an assembly line are inspected, and each is classified as either conforming (acceptable), downgraded, or rejected. Overall, 70% of the items are conforming,① 20% are downgraded,② and 10% are rejected.③ Assume that four items are chosen independently and at random. Let X_1, X_2, X_3 denote the numbers among the 4 that are conforming, downgraded, and rejected, respectively.

3 possible outcome var.
↓
Burdan anladık multinomial olduğunu

X_1 : # of item conforming → prob → 0,7
X_2 : # of item downgraded → prob → 0,2
X_3 : # of item rejected → prob → 0,1

1. What is the distribution of X_1, X_2, X_3?
2. What is the probability that 3 are conforming and 1 is rejected in a given sample?

1) $X_1, X_2, X_3 \sim MN(4, 0.7, 0.2, 0.1)$

2) $P(X_1 = 3, X_2 = 0, X_3 = 1) = \frac{4}{3! \cdot 0! \cdot 1!} \cdot (0,7)^3 \cdot (0,2)^0 \cdot (0,1)^1 = 0,1372$

Ex

9 kişi bir toplantıya katılmak için uçak (P_1), otobüs (P_2), otomobil (P_3) ve tren (P_4) seçeneklerini sırasıyla 40%, 20%, 30%, 10% oranlarıyla tercih edebilmektedirler. Buna göre bu 9 kişiden 3'ünün uçağı, 3'ünün otobüsü, 1 kişinin otomobili ve 2 kişinin treni tercih etme olasılığını bulunuz.

$P_1 \to 0,4 \to 3$
$P_2 \to 0,2 \to 3$
$P_3 \to 0,3 \to 1$
$P_4 \to 0,1 \to 2$
$\quad\;\; + \quad\; +$
$\quad\;\; 1 \quad\;\; 9$

$$\frac{9!}{3! \cdot 3! \cdot 1! \cdot 2!} \cdot (0,4)^3 \cdot (0,2)^3 \cdot (0,3)^1 \cdot (0,1)^2$$

//

HYPERGEOMETRIC DISTRIBUTION

→ Bernoulli deneyidir. İki olası sonuç içeren durumlarda kullanılır.

→ Deneyler independent değillerdir. → Because of sampling without replacement.

↳ Independent olmadığı için Binomial Distribution diyemeyiz.

↳ Böyle bir olayda, number of success'i tanımlayan duruma Hypergeometrik Distribution denir.

* Kombinasyon yardımıyla olasılık hesaplanır.

$$p(x) = P(X=x) = \begin{cases} \boxed{\dfrac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}}}, & ---- \\ 0, & \text{otherwise} \end{cases}$$

N = total item
R = # of success
n = sampled from population
X = # of success

→ $X \sim H(N,R,n)$

$$\mu_x = E(x) = \frac{nR}{N}$$

$$\sigma^2 = n \cdot \left(\frac{R}{N}\right)\left(1-\frac{R}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Ex

Bir torbada $\overset{N}{20}$ top vardır. Bu toplardan $\overset{R}{12}$ tanesi kırmızıdır.

a) Geri bırakılmaksızın seçilen $\underset{n}{8}$ toptan $\underset{x}{5}$'inin kırmızı olma olasılığı nedir?

b) Geri bırakılmaksızın seçilen $\underset{n}{4}$ toptan en çok 1 tanesinin kırmızı olma olasılığı nedir?

x ← 0 kırmızı – 4 k. olmayan
veya → n-x
x ← 1 kırmızı – 3 k. olmayan

a) $\dfrac{\binom{12}{5}\binom{8}{3}}{\binom{20}{8}}$

b) $\dfrac{\binom{12}{0}\binom{8}{4}+\binom{12}{1}\binom{8}{3}}{\binom{20}{4}}$

Ex

İçinde $\overset{R}{10}$ adet üçgen $\overset{N-R}{5}$ adet kare oyuncak içeren bir torbadan $\underset{n}{3}$ adet oyuncak geri konulmaksızın alınıyor.

a) 3'ünün de üçgen olan oyuncak olma olasılığı nedir?
$\underbrace{3}_{x}$

b) En az 2 kare oyuncak gelme olasılığı nedir?
$\underbrace{2}_{n-x}$

a) $$\frac{\binom{10}{3}\binom{5}{0}}{\binom{15}{3}}$$

b) $$\frac{\binom{10}{1}\binom{5}{2}+\binom{10}{0}\binom{5}{3}}{\binom{15}{3}}$$

Ex

Of 50 buildings in an industrial park, 12 have electrical code violations. If 10 buildings are selected at random for inspection, what is the probability that exactly 3 of the 10 have code violations? What are the mean and variance of X?

$N=50 \quad n=10$
$K=12 \quad x=3$

$$=\frac{\binom{12}{3}\binom{38}{7}}{\binom{50}{10}}$$

$$\mu = E(x) = \frac{nK}{N} = \frac{10.12}{50} = 2,4$$

$$\sigma^2 = 10.\left(\frac{12}{50}\right)\left(1-\frac{12}{50}\right)\left(\frac{50-10}{50-1}\right)$$

→ **GEOMETRIC DISTRIBUTION**

→ 2 olası sonuç olan durumlarda Geometrik dağılım sorulabilir.

→ ilk başarının n.'de görülme olasığını hesaplamada kullanılır.

* p: başarı olasılığı
q: başarısızlık olasılığı
$\boxed{p+q=1}$

$\boxed{X \sim Geom(p)}$

* İlk başarının n.de görülme olasılığı ⇒ ilk n-1 deneme başarısız demektir

$$\boxed{p(x) = P(X=x) = p.\,q^{n-1}}$$

Ex

Bir hastanede bir hastalıktan iyileşme olasılığı 0,05'tir. Bu hastaneye gelenlerden ilk iyileşenin 8. hasta olma olasılığı kaçtır?

$\hookrightarrow n-1=7$ (ilk 7 hasta iyileşmemiş)

$= (0,95)^7 (0,05)$

Ex

Bir zarın atılması deneyinde ilk kez 5 gelmesinin 4. denemede gerçekleşme olasılığı nedir?

$$\left.\begin{matrix} p = 1/6 \\ q = 5/6 \end{matrix}\right\} \quad \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$$

* $\boxed{\mu_x = E(x) = \frac{1}{p}}$

* $\boxed{\sigma_x^2 = \frac{1-p}{p^2}}$

Ex

A test of weld strength involves loading welded joints until a fracture occurs. For a certain type of weld, 80% of the fractures occur in the weld itself, while the other 20% occur in the beam. A number of welds are tested. Let X be the number of tests up to and including the first test that results in a beam fracture.

probability of beam fracture = success = 0,2
probability of failure = 0,8

1. What is the distribution of X? → $X \sim Geom(p)$
$X \sim Geom(0,2)$
2. Find $P(X = 3)$. $= (0,2)(0,8)^{3-1} = (0,2)(0,8)^2$
3. What are the mean and variance of X?

$\mu = E(x) = \frac{1}{p} = \frac{1}{0,2} = 5$

$\sigma^2 = \frac{1-p}{p^2} = \frac{q}{p^2} = \frac{1-0,2}{(0,2)^2} = \frac{0,8}{0,04} = 20$

NEGATIVE BINOMIAL DISTRIBUTIONS

→ x. başarının n. denemede gerçekleşme olasılığını hesaplamada kullanılır.

↳ n. deneme kesinlikle başarıdır.

↳ İlk n−1 deneme için binom söz konusudur.

* $\boxed{\left[\binom{n-1}{x-1}\cdot p^{x}\cdot q^{n-x}\right]}$ → $\boxed{X \sim NB(x,p)}$

* $\boxed{\mu_x = \dfrac{x}{p}}$ * $\boxed{\sigma_x^2 = \dfrac{x\cdot(1-p)}{p^2} = \dfrac{x\cdot q}{p^2}}$

Ex

Bir hastanede bir hastalıktan iyileşme olasılığı 0,03'tür. Bu hastaneye gelen hastalardan 5. iyileşen kişinin gelenlerden 12.'si olma olasılığı kaçtır?

↳ 12. kişi iyileşen

↳ ilk 11 için BİNOM

$$\left[\binom{11}{4}(0,03)^5(0,97)^7\right]$$

Ex

In a test of weld strength, 80% of tests result in a fracture in the weld, while the other 20% result in a fracture in the beam. Let X denote the number of tests up to and including the third beam fracture.

a) What is the distribution of X? → $X \sim NB(x,p)$, $X \sim NB(3,\ 0.2)$

b) Find $P(X=8)$. $= \left[\binom{8-1}{3-1}(0,2)^{3}\cdot(0,8)^{8-3}\right]$

↳ 8. olma olasılığı

c) Find the mean and variance of X.

$\mu_x = \dfrac{x}{p} = \dfrac{3}{0,2} = 15$ $\quad \sigma_x^2 = \dfrac{x\cdot q}{p^2} = \dfrac{3\cdot(0,8)}{(0,2)^2} = \dfrac{3\cdot(0,8)}{(0,04)} = 60$

→ **POISSON DISTRIBUTION**

↳ Discrete'dir. !

* Nadir gerçekleşen olayların olasılığının hesaplanmasında kullanılır.

* Nadir gerçekleşen olayların olasılığının hesaplanmasında kullanılır.

→ Soruda mutlaka nadir gerçekleşen bu olayın belli bir zaman dilimi veya belli bir bölge için gerçekleşme ortalaması verilmiş olmalıdır.

* $\lambda \to$ soruda verilen ortalama
* $x \to$ istenen olasılık adedi

$$\boxed{P(x) = P(X=x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}}$$

→ e : euler sayısı = 2,71...

$$\boxed{\begin{aligned} \mu_x &= E(x) = \lambda \\ \sigma_x^2 &= \lambda \end{aligned}}$$

!

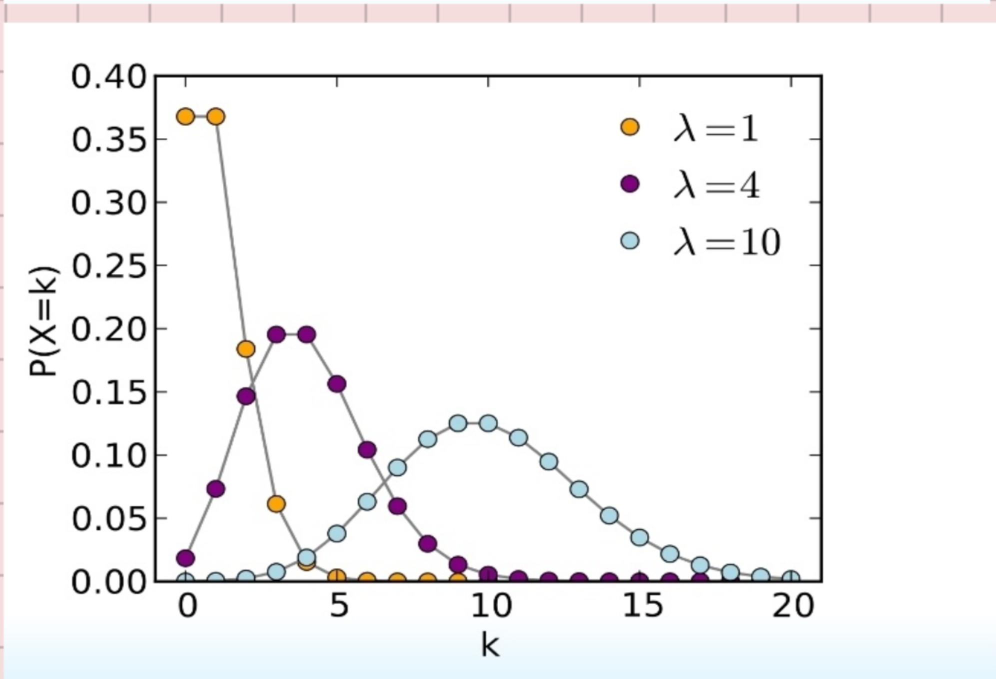
* Beşiktaş ilçesinde 3 aylık sürede yangın ortalaması 7'dir. → $\lambda = 7$ (3 ay) → süre değişirse λ değişir.

- X: the number of events occurred in a fixed interval of time or space
- The number of births per hour during a given day
- The number of particles emitted by a radioactive source in a given time
- The number of cases of a disease in different towns
- The number of hits on a web site in one hour
- The number of goals scored by a football team in a match
- The number of accidents in a certain part of the road
- The number of customers who enters a market during an hour

→ One way to think of the Poisson distribution is as an aproximation to the binomial distribution when n is large and p is small;

$\lambda = np$

$X \sim \text{Poisson}(\lambda)$



Ex

Belirli bir kavşakta 4 aylık sürede kaza ortalaması 6'dır.

a) Buna göre, bu kavşakta gelecek 4 ayda 7 kaza olma olasılığı nedir?

b) Buna göre, bu kavşakta gelecek 6 ayda 5 kaza olma olasılığı nedir?

c) Buna göre, bu kavşakta gelecek 1 yılda 2'den fazla kaza olma olasılığı?

d) Buna göre, gelecek 8 aylık sürede en çok 2 kaza olma olasılığı nedir?

a) $\lambda = 6$ (4 ay), $x = 7$

$$\frac{6^7 \cdot e^{-6}}{7!}$$

b) $\lambda = 6$ (4 ay) $\longrightarrow$ 4 ay 6, 6 ay ?

$\lambda = 9$ (6 ay)

$x = 5$

$$\frac{9^5 \cdot e^{-9}}{5!}$$

c) $\lambda = 6$ (4 ay)

$\lambda = 18$ (12 ay)

$x = 3, 4, 5, \ldots$

$\hookrightarrow 1 - (x = 0, 1, 2)$

1 - istenmeyen

$$1 - \frac{18^0 \cdot e^{-18}}{0!} - \frac{18^1 \cdot e^{-18}}{1!} - \frac{18^2 \cdot e^{-18}}{2!}$$

d) $\lambda = 6$ (4 ay)

$\lambda = 12$ (8 ay)

$x = 0, 1, 2$ kaza

$$\frac{12^0 \cdot e^{-12}}{0!} + \frac{12^1 \cdot e^{-12}}{1!} + \frac{12^2 \cdot e^{-12}}{2!}$$

Ex

X rastgele değişkeni ortalama değeri 4 olan bir Poisson dağılımına sahip olsun. ($\lambda = 4$)

Buna göre aşağıdaki olasılıkları hesaplayınız.

a) $P(2 \le x \le 5)$

b) $P(x \geq 3)$

c) $P(x \le 3)$

a) $P(x=2) + P(x=3) + P(x=4) + P(x=5)$

$$\frac{4^2 \cdot e^{-4}}{2!} + \frac{4^3 \cdot e^{-4}}{3!} + \frac{4^4 \cdot e^{-4}}{4!} + \frac{5^4 \cdot e^{-5}}{5!}$$

b) $P(x=3) + P(x=4) + \ldots = 1 - P(x=2) - P(x=1) - P(x=0)$

$$1 - \frac{4^2 \cdot e^{-4}}{2!} - \frac{4^1 \cdot e^{-4}}{1!} - \frac{4^0 \cdot e^{-4}}{0!}$$

c) $P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$$\frac{4^0 \cdot e^{-4}}{0!} + \frac{4^1 \cdot e^{-4}}{1!} + \frac{4^2 \cdot e^{-4}}{2!} + \frac{4^3 \cdot e^{-4}}{3!}$$

Ex

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter. Determine the probability of exactly two flaws in 1 millimeter of wire.

X : number of flaws

λ : average number of flaws = 2.3

$$P(x=2) = P(2) = \frac{e^{-2,3} \cdot (2,3)^2}{2!} = 0,2652$$

Ex

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter. Determine the probability of 10 flaws in 5 millimeters of wire.

$\lambda = 2.3$ (for 1mm) $\rightarrow$ 1mm $\lambda = 2.3$

5mm $\lambda = ?$

$\lambda = 2.3 \times 5 = 11,5$

$$P(x=10) = \frac{e^{-11,5} \cdot (11,5)^{10}}{10!} = 0,7358 \cdot 10^{-6}$$

Ex

Grandma bakes chocolate chip cookies in batches of 100. She puts 300 chips into the dough. When the cookies are done, she gives you one.

What is the probability that your cookie contains no chocolate chips?

X = the number of chips

λ = average number of chips per cookie

$\lambda = \frac{300}{100} = 3$

$$P(x=0) = \frac{e^{-3} \cdot 3^0}{0!} = 0,0498$$

Ex

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away? → 15'i aşarsa

Let X be the number of tankers arriving each day. Then, using Table A.2, we have

$$P(X > 15) = 1 - P(X \leq 15) = 1 - \sum_{x=0}^{15} p(x; 10) = 1 - 0.9513 = 0.0487.$$

↳ $\lambda = 10$

→ SOME CONTINOUS PROBABILITY DISTRIBUTIONS

→ CONTINOUS UNIFORM DISTRIBUTION

rastgele değişken

→ X için belli bir aralıkta düzgün dağıldığı söylenirse :

* For Example ; X rastgele değişkeni (2,5) aralığında düzgün dağılıma sahiptir.

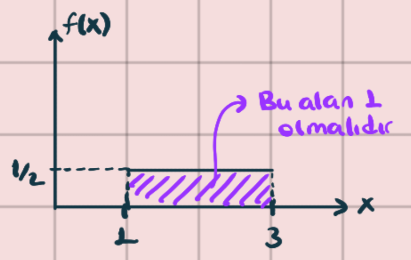
* $X \sim U(a,b)$

* X, (a,b) aralığında düzgün dağılıma sahip ise :

$$\boxed{f(x) = \begin{cases} \dfrac{1}{a-b}, & a<x<b \\ 0, & \text{otherwise} \end{cases}}$$

* $\boxed{\mu_x = \dfrac{a+b}{2}}$

* $\boxed{\sigma_x^2 = \dfrac{(b-a)^2}{12}}$



* The density function for a random variable on the interval 1,3

Ex

X rastgele değişkenine sahip bir olasılık dağılımı (3,7) aralığında düzgün dağılıma sahiptir. Buna göre ;

a) $P(x=2) = ?$ 0 (f(x) süreklidir) (Soruda istenen discrete o yüzden 0 olur)

b) $P(4 < x \leq 5) = ?$ → $\int_4^5 1/4\,dx = 1/4$

c) $P(x < 4) = ?$ → $\int_{-\infty}^4 1/4\,dx = \int_3^4 1/4\,dx = 1/4$

d) $P(4 \leq x) = ?$ → $\int_4^\infty 1/4\,dx = \int_4^7 1/4\,dx = 3/4$

sürekli

$$\boxed{f(x) = \begin{cases} 1/4, & 3<x<7 \\ 0, & \text{otherwise} \end{cases}}$$

Ex

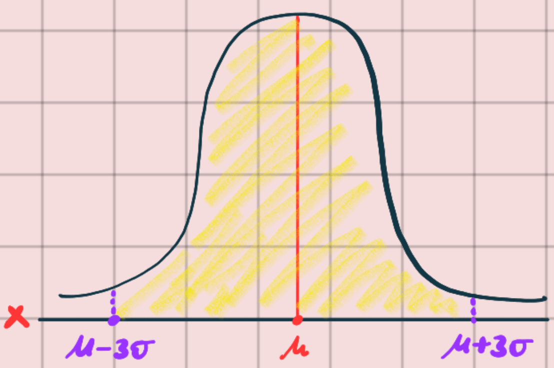
When a motorist stops at a red light at a certain intersection, the waiting time for the light to turn green, in seconds, is uniformly distributed on the interval (0, 30). Find the probability that the waiting time is between 10 and 15 seconds.

$X \sim U(0,30)$

$$f(x) = \begin{cases} \frac{1}{30}, & 0<x<30 \\ 0, & \text{otherwise} \end{cases}$$

$$P(10<x<15) = \int_{10}^{15} \frac{1}{30}\,dx = \frac{15}{30} - \frac{10}{30} = \frac{5}{30} = \frac{1}{6}$$

→ NORMAL DISTRIBUTIONS



① Çan eğrisinin altında kalan alan 1'dir.

② Maksimum nokta μ ile eşleşir.

③ Simetriktir.

④ μ+3σ ve μ−3σ sağa ve sola gidildiğinde tüm alanın %99,9.... civarlarına ulaşılır.

* $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\boxed{-\infty < x < \infty}$

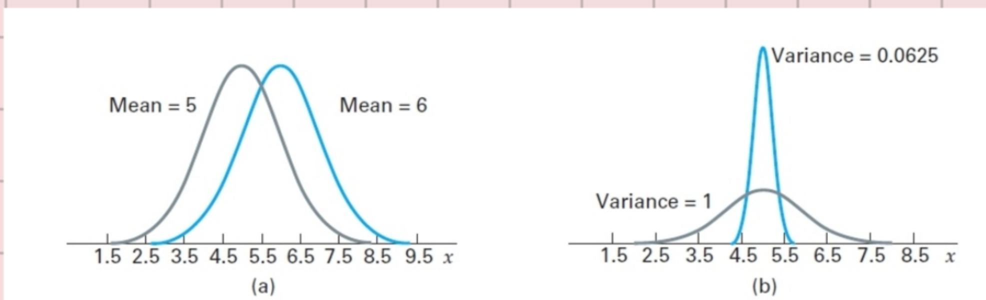
* $\mu_x = \mu$

* $\sigma_x^2 = \sigma^2$

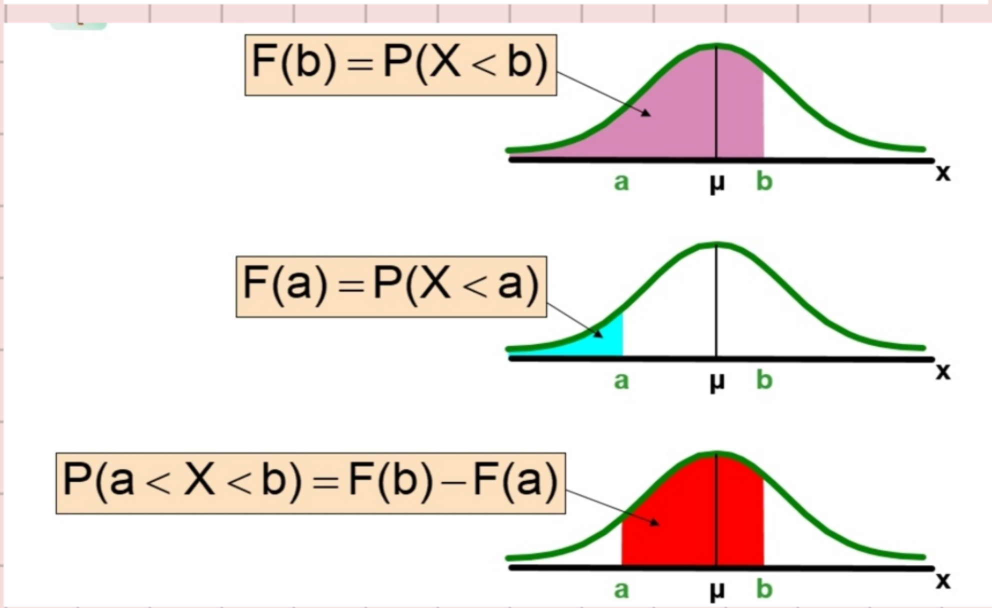
* $X \sim N(\mu, \sigma^2)$

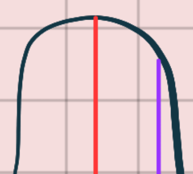
* $\int_{-\infty}^{\infty} f(x)\,dx = 1$

* $f(x) \geq 0$

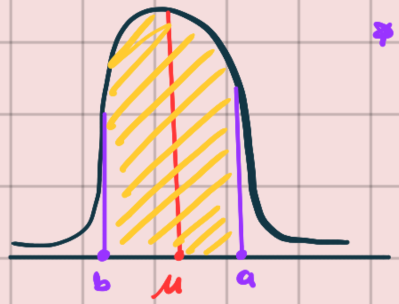


a. Two Normal Distributions with Same Variance but Different Means
b. Two Normal Distributions with Different Variances and Mean = 5





* P(x=a) = ? 0 → (Normal distribution süreklidir. Aralık bildiren olasılıkların değerleri vardır. Eşitlik bildirenler 0'a eşittir.)



* $P(a < x \leq b) = ?$

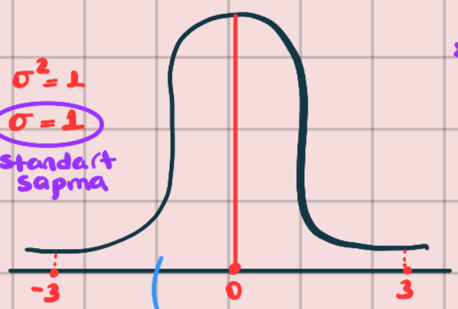
$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

* Bu integralin hesaplanması zor olduğu için, bizim yerimize bu integraller hesaplanıp bir z tablosu oluşturulmuştur.

* Normal Dağılım (x'e bağlıdır) $\xrightarrow{\text{Normal Dağılım, Standart ND'ye çevrilir.}}$ * Standart Normal Dağılım (z'ye bağlıdır) → z tablosu kullanılarak olasılık hesaplanır.

→ STANDART NORMAL DISTRIBUTION

$$\boxed{z = \frac{x - \mu}{\sigma}}$$ → Standartlaştırma işlemi



* Ortalaması 0 ve varyansı 1 haline getirilen normal dağılımlara standart normal dağılım denir.

(Standart Normal Population)

Bunun altında kalan alanları z tablosu ile hesaplarız.

Ex

$\underbrace{\text{Ortalaması 6}}_{\mu}$ ve $\underbrace{\text{standart sapması 2}}_{\sigma}$ olan bir normal dağılımda $P(x<5) = ?$

$P(x<5) \Rightarrow P\left(z < \frac{5-6}{2}\right) = P(z < -0{,}5)$

$z = \frac{x-\mu}{\sigma}$

Taralı alan bu olasılığı verecek. Hesaplanması için de z tablosu kullanılır.

Ex

Aluminum sheets used to make beverage cans have thicknesses (in thousandths of an inch) that are normally distributed with mean 10 and standard deviation 1.3.

a) A particular sheet is 10.8 thousandths of an inch thick. Find the z-score.

b) The thickness of a certain sheet has a z-score of -1.7. Find the thickness of the sheet in the original units of thousandths of inches.

$X \sim N(10, (1.3)^2)$

a) $z = \dfrac{x-\mu}{\sigma}$ $\qquad z = \dfrac{10.8-10}{1.3} = \dfrac{0.8}{1.3} = 0.615$

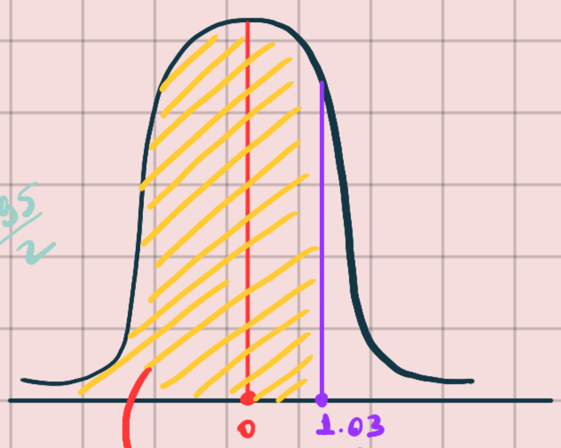
b) $z = \dfrac{x-10}{1.3} = -1.7$ $\qquad x = 1.3 \times (-1.7) + 10$

$x = 12.21$

→ **AREAS UNDER THE NORMAL CURVE**

Standard Normal Distribution
(Values of Cumulative Distibution Function F(z))

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990



$1.0 + 0.03 = 1.03$

Bir tablo çizildiğinde ve bir nokta seçildiğinde o nokta tabloyu ikiye böler. Böldüğü kısımlardan büyük olanın alanını bu z tablosu verir.

Ex

a) Find the area under normal curve to the left of $z = 0.47$.

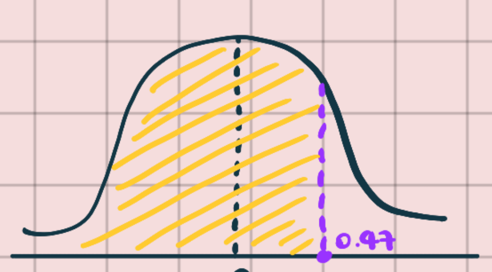
b) Find the area under the curve to the right of $z = 1.38$.

c) Find the area under the curve to the left of $z = -1.55$.

d) Find the area under the normal curve between $z = 0.71$ and $z = 1.28$.

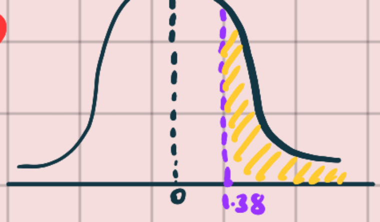
e) What z-score corresponds to the 75[th] percentile of a normal curve?

a)



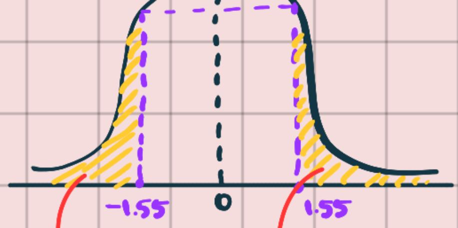
$P(z < 0.47) = F(0.47) = 0.6808$

b)



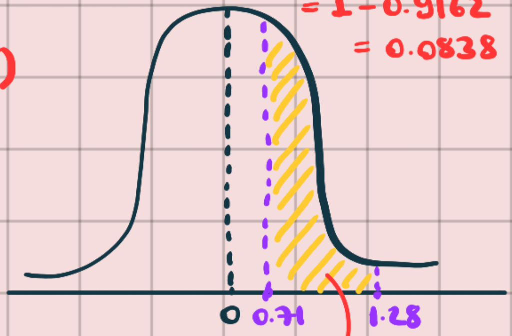
$P(z > 1.38) = 1 - P(z < 1.38)$
$= 1 - F(1.38)$
$= 1 - 0.9162$
$= 0.0838$

c)



$P(z < -1.55) = P(z > 1.55) = 1 - F(1.55)$

d)

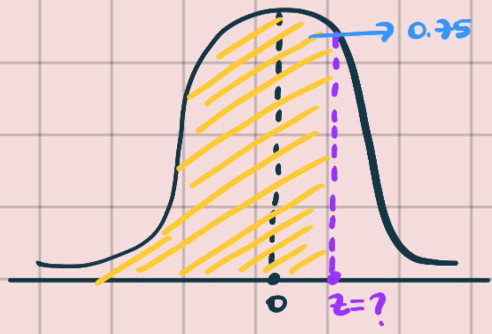


$= P(0.71 < z < 1.28)$

$= F(1.28) - F(0.71)$

$= 0.8997 - 0.7611$

e)



$P(z < Z) = 0.75 \Rightarrow F(z) = 0.75$

$z = 0.6745$

<u>Ex</u>

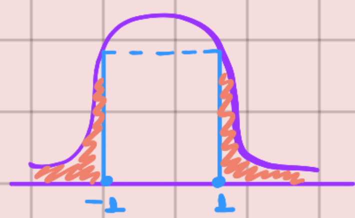
A random variable has a Normal distribution with mean 69 and standard deviation 5.1. What are the probabilities that the random variable will take a value

a) less than 74.1?
b) greater than 63.9?
c) between 69 and 72.3?
d) between 66.2 and 71.8?

a) $P(x < 74.1) = P\left(\frac{x-\mu}{\sigma} < \frac{74.1-69}{5.1}\right)$

$= P(z < 1) = F(1) = 0.8413$

b) $P(x > 63.9) = P\left(\frac{x-\mu}{\sigma} > \frac{63.9-69}{5.1}\right)$



$= P(z > -1) = P(z < 1) = F(1) = 0.8413$

c) $P(69 < x < 72.3) = P\left(\frac{69-69}{5.1} < z < \frac{72.3-69}{5.1}\right)$

$= P(0 < z < 0.6471) = F(0.65) - F(0)$

$= P(0 < z < 0.6471) = F(0.65) - F(0)$

0.65

Ex

Bir sınıftaki öğrencilerin boylarının uzunluğu normal dağılmaktadır. Bu sınıftaki öğrencilerin boylarının uzunluğunun ortalaması 160 cm ve standart sapması 5 cm dir.

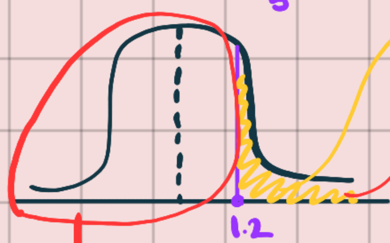
standart normal dağılıma çevrilmeli

Bu sınıftan seçilen bir öğrencinin boyunun 166 cm'den uzun olma olasılığı kaçtır?

$\mu = 160$ cm

$\sigma = 5$ cm

$P(x > 166) = ?$ → $P\left(z > \frac{166-160}{5}\right) = P(z > 1.2)$



→ $1 - 0.8849 = 0.1151$

0.8849
(z tablosu burayı verir.)

Ex

Bir yoldan geçen araçların hızları normal dağılmaktadır. Bu yoldan geçen araçların ortalama hızı 65 km/sa ve standart sapması 2 km/sa'tir.

Bu yolda belli bir hızın aşılması durumunda araçlara ceza kesildiğine göre, bu yolda izin verilen en yüksek hız kaç km/sa'tir?

$\mu = 65$

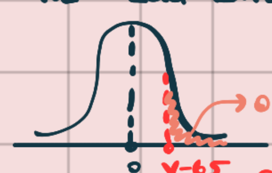
$\sigma = 2$

$z = \frac{x - \mu}{\sigma}$

V = ceza kesilen hız?

$P(x > V) = 0.25$

$P\left(z > \frac{V-65}{2}\right) = 0.25$



$\frac{V-65}{2} = 0.675 = \frac{0.67 + 0.68}{2}$

(Tablodan yaklaşık olan iki değerin ort.'unu aldık)

$V - 65 = 1.350$

$\boxed{V = 66.35 \text{ km/sa}}$

Ex

Lifetimes of batteries in a certain application are Normally distributed with mean 50 hours and standard deviation 5 hours.

Find the probability that a randomly chosen battery lasts between 42 and 52 hours.

$\mu = 50$

$\sigma = 5$

$P(42 < x < 52) = ?$

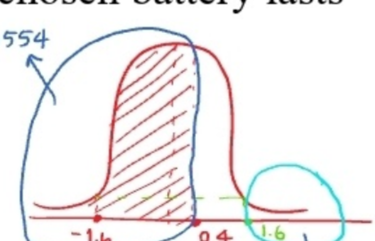
$P\left(\frac{42-50}{5} < z < \frac{52-50}{5}\right)$

$P(-1.6 < z < 0.4)$

$P(z < 0.4) - P(z > 1.6)$

$= 0.6554 - 0.0548$

$= 0.6006$



$1 - 0.9452$
$= 0.0548$

Ex

A random variable has a Normal distribution with variance 100. Find its mean if the probability that it will take on a value less than 77.5 is 0.8264.

$\sigma^2 = 100$

$\sigma = 10$

$P(X < 77.5)$

$P\Big(Z < \underbrace{\frac{77.5-\mu}{10}}_{0.94}\Big) = 0.8264$

$\frac{77.5-\mu}{10} = 0.94$

$\boxed{\mu = 68.1}$

Ex

A process manufactures ball bearings whose diameters are normally distributed with mean 2.505 cm and standard deviation 0.008 cm. Specifications call for the diameter to be in the interval 2.5 ± 0.01 cm. What proportion of the ball bearings will meet the specification?

$X \sim N(2.505,\ 0.008^2)$

$P(\underbrace{2.49 < X < 2.51}_{2.50 \mp 0.01}) = P\left(\frac{2.49-2.505}{0.008} < X < \frac{2.51-2.505}{0.008}\right)$

Ex

Gauges are used to reject all components for which a certain dimension is not within the specification 1.50±*d*. It is known that this measurement is Normally distributed with mean 1.50 and standard deviation 0.2. Determine the value *d* such that the specifications cover 95% of the measurements.

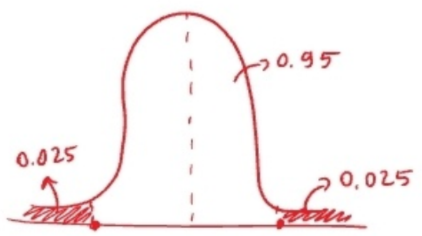
$\mu = 1.50$

$\sigma = 0.2$

$P(Z<\ldots) - (1 - P(Z<\ldots)) = 0.95$

$-1 + 2P(Z<\ldots) = 0.95$

$2\underbrace{P(Z<\ldots)}_{1.96} = 1.95$



$P(-1.96 < Z < 1.96) = 0.95$

$1.96 = \frac{(1.50+d) - 1.50}{0.2}$

$\boxed{d = 0.392}$

→ EXPONENTIAL DISTRIBUTION

→ Zamansal olasılık dağılımı sorularında karşımıza çıkar.
(Bütün üstel dağılım soruları zamanla ilgidir.)

* μ (ortalama) (mean) → soruda verilir.
↳ * İki olayın gerçekleşmesi arasında geçen sürelerin ortalaması

* Bir banka veznesinde iki müşteri arasında geçen sürenin ortalaması 3 dakikadır.

* Bir duraktan iki otobüsün arka arkaya geçmesi için gereken ortalama süre 7 dakikadır. ($\mu = 7$)

pdf of an exponential

* $$f(x) = \begin{cases} \lambda . e^{-\lambda x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

* $\boxed{\mu_x = 1/\lambda}$

* $\boxed{\sigma_x^2 = 1/\lambda^2}$

* $\boxed{X \sim Exp(\lambda)}$

* Sürekli bir dağılım olduğundan olasılıklar integral yardımıyla hesaplanır.

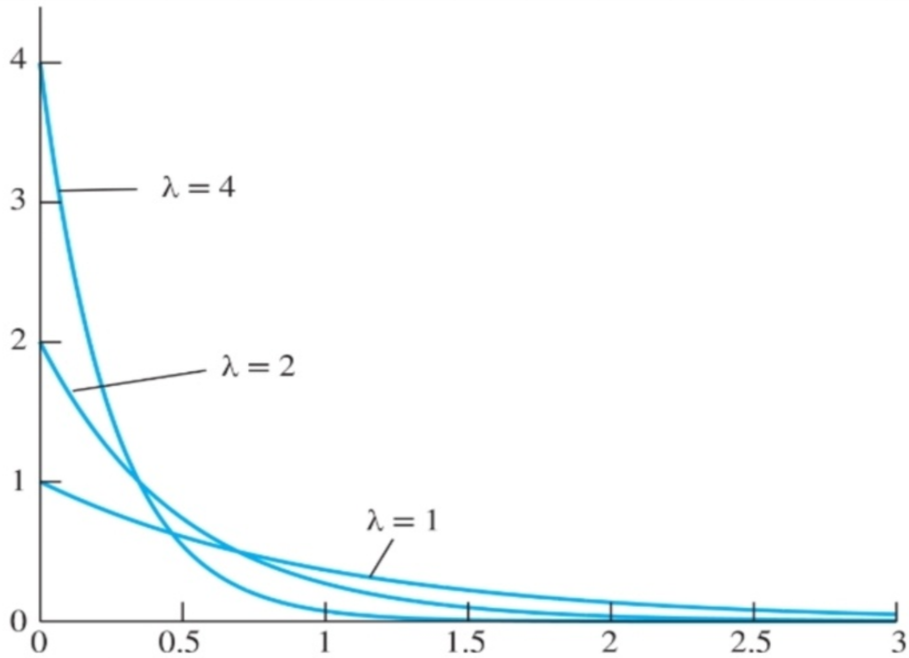


FIGURE 4.17 Plots of the exponential probability density function for various values of λ.

Ex

Bir duraktan otobüslerin geçme süreleri üstel dağılıma sahiptir. Arka arkaya iki otobüsün geçmesi için gerekli ortalama zaman 5 dakikadır.

a) Bu durağa gelen bir kişinin en fazla 3 dakika bekleme olasılığı?

b) Bu durağa gelen bir kişinin 6 dakikadan fazla bekleme olasılığı?

c) Bu durağa gelen bir kişinin 8 dakika bekleme olasılığı?

d) Beklenen değer ve varyans?

$$\boxed{\int e^{at}\,dt = \frac{e^{at}}{a} + C}$$

a) $\mu = 5 = 1/\lambda$ $\quad$ $\lambda = 1/5$

$$f(x) = \begin{cases} \frac{1}{5} e^{-x/5}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$P(x \leq 3) = ?$

$$\int_{-\infty}^{3} \frac{1}{5} e^{-x/5}\,dx = \int_{0}^{3} \frac{1}{5} e^{-x/5}\,dx = \left.\frac{\frac{1}{5} e^{-x/5}}{-\frac{1}{5}}\right|_0^3 = 1 - e^{-3/5}$$

b) $P(x \geq 6) = ?$

$$\int_{6}^{\infty} \frac{1}{5} e^{-x/5}\,dx = -e^{-x/5}\Big|_6^{\infty} = -e^{(-\infty)} - (-e^{-6/5}) = e^{-6/5}$$

$$\boxed{\begin{array}{l} e^{-\infty} = 0 \\ e^{\infty} = \infty \end{array}}$$

c) $P(x = 8) = ? \rightarrow 0$

d) $\mu = 5$ $\quad$ $\sigma^2 = 25$

!

$$\boxed{F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}}$$

cumulative density (cdf) function of an exponential

Ex

Example 4.56

If $X \sim \text{Exp}(2)$, find μ_X, σ_X^2, and $P(X \leq 1)$.

$\lambda = 2$

$\mu = 1/\lambda = 1/2$

$\sigma^2 = 1/\lambda^2 = 1/4$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

$P(X \leq 1) = F(1) = 1 - e^{-2} = 1 - 0.1353 = 0.8647$

→ The Exponential Distribution and the Poisson Process

→ If events follow a Poisson process with rate parameter λ and if T represents the waiting time from any starting point until the next event, then $T \sim Exp(\lambda)$

* The probability density function (PDF) of an exponential distribution is given by:

$f(t) = \lambda \cdot e^{-\lambda t}$

→ t is waiting time

→ λ is the rate parameter (average rate of events per unit time)

Ex

A radioactive mass emits particles according to a Poisson process at a mean rate of 15 particles per minute. At some point, a clock is started.

1. What is the probability that more than 5 seconds will elapse before the next emission?

→ This is the same as finding the probability that no emmission occurs within the first 5 seconds

2. What is the mean waiting time until the next particle is emitted? $\mu = \frac{1}{\lambda} = \frac{1}{0.25} = 4$ seconds

1) 60 seconds — 15 particles
1 seconds — ?

$15/60 = 0.25$

$X \sim Poisson(\lambda = 15 \text{ per min})$

$T \sim$ waiting time $(\lambda' = 0.25 \text{ per sec.})$

P(x > 5 seconds) = ?

$T \sim Exp(0.25) \rightarrow P(X>5) = 1 - P(X<5)$

$= 1 - (1 - e^{-0.25 \cdot 5}) = e^{-1.25}$

Absent of something

→ Lack Of Memory Property

→ If $T \sim Exp(\lambda)$, and t and s are positive numbers, then

$P(T>t+s \mid T>s) = P(T>t)$

* $\frac{e^{-(t+s)\cdot\lambda}}{\ } = \cancel{e^{-s\lambda}} \cdot e^{-t\lambda}$

$e^{-s \cdot \lambda}$ $\cancel{e^{-s\lambda}}$ $\rightarrow$ $P(T>t)$

* $\dfrac{P(T>s,\ T>t+s)}{P(T>s)} \longrightarrow \dfrac{P(T>t+s)}{P(T>s)} \rightarrow P(T>t)$

Ex

The lifetime of a particular integrated circuit has an exponential distribution with mean 2 years. Find the probability that the circuit lasts longer than three years.

$\mu = 2$

$\mu = \frac{1}{\lambda} = 2 \rightarrow \lambda = 1/2$

$T \sim$ lifetime of something

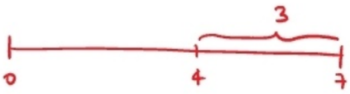
$T \sim Exp(\lambda = 1/2)$

$\boxed{P(T>3) = e^{-1/2 \cdot 3}} = e^{-3/2} \approx 0.2231$

* $P(T<t) = 1 - e^{-\lambda \cdot t}$
* $P(T>t) = e^{-\lambda \cdot t}$

Ex

Refer to Example 4.59. Assume the circuit is now four years old and is still functioning. Find the probability that it functions for more than three additional years. Compare this probability with the probability that a new circuit functions for more than three years, which was calculated in Example 4.59.



$P(T>7 \mid T>4) = P(T>3)$ $\quad F(x) = 1 - e^{-\lambda x}$

$P(T>3) = 1 - P(T \leq 3) = 1 - F(3)$

$P(T>3) = 1 - F(x) = 1 - (1 - e^{-1/2 \cdot 3}) = e^{-3/2}$

Ex

The number of hits on a website follows a Poisson process with a rate of 3 per minute.

1. What is the probability that more than a minute goes by without a hit? → This is the same as finding the probability that no hits occur in the first minute.

2. If 2 minutes have gone by without a hit, what is the probability that a hit will occur in the next minute?

1) $P(x=0) = \dfrac{e^{-\lambda t} \cdot (\lambda t)^0}{0!}$ $\quad \lambda = 3$ (rate per minute)

$t = 1$ minute

$P(\text{no hit in 1 minute}) = e^{-3 \cdot 1} = e^{-3} = 0.0498$

$P(\text{no hit more than 1 minute}) = 1 - 0.0498 = 0.9502 \rightarrow 95.02\%$

2)

$P(T>3 \mid T>2) = P(T>1) = 1 - P(T \le 1) = 1 - F(1)$

$= 1 - (1 - e^{-\lambda \cdot t}) = 1 - (1 - e^{-3 \cdot 1}) = e^{-3} = 0.0498$