June 22, 2022

EEE 208 ELECTRIC CIRCUIT ANALYSIS II
EEE 202 ELECTRIC CIRCUITS II

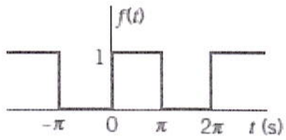
SPRING'22 - FINAL EXAM

SOLUTIONS

Name: ______________________, **Signature:** ______________________

Closed books and closed notes. Use of mobile phones is considered cheating. Show your work clearly in the spaces provided. Otherwise, no credit will be given.

(20 pts) **Q.1** Find the Fourier series expression for the periodic function shown in the following figure.



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n w_0 t + \sum_{n=1}^{\infty} b_n \sin n w_0 t$$

$$T = 2\pi, \quad w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi}\ 1 \text{ rad/s}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} dt = \frac{1}{2\pi} t \Big|_0^{\pi} = \frac{1}{2}$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = \frac{1}{\pi}\frac{\sin nt}{n}\Big|_0^{\pi} = 0$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)\Big|_0^{\pi}$$

$$= \frac{1}{n\pi}(1 - \cos n\pi)$$

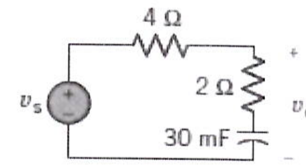
$b_n = 0$ if n is even,
$b_n = \frac{2}{n\pi}$ if n is odd.

Thus, as the expression of the Fourier series

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)t$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

(20 pts) **Q.2** For the circuit shown in the following figure:



a) Determine the network function $\mathbf{H}(\omega) = \mathbf{V}_o/\mathbf{V}_s$.

$$\frac{\bar{V}_o(w)}{\bar{V}_s(w)} = \frac{2+\frac{1}{jw\,30\times10^{-3}}}{4+2+\frac{1}{jw\,30\times10^{-3}}} = \frac{2+\frac{100}{j3w}}{6+\frac{100}{j3w}} = \frac{100+j6w}{100+j18w}$$

$$= \frac{1+j\frac{w}{100/6}}{1+j\frac{w}{100/18}} = \frac{1+jw/16.7}{1+jw/5.56} = \frac{1+jw/w_z}{1+jw/w_p}$$

$$\bar{H}(w) = \frac{1+jw/16.7}{1+jw/5.56} = \frac{\sqrt{1+w^2/16.7^2}}{\sqrt{1+w^2/5.56^2}} \angle \tan^{-1}(w/16.7) - \tan^{-1} w/5.56)$$

b) Sketch the asymptotic magnitude Bode plot for the network function $\mathbf{H}(\omega)$.

$$\text{Gain} = |\bar{H}(w)| = \frac{\sqrt{1+w^2/16.7^2}}{\sqrt{1+w^2/5.56^2}}$$

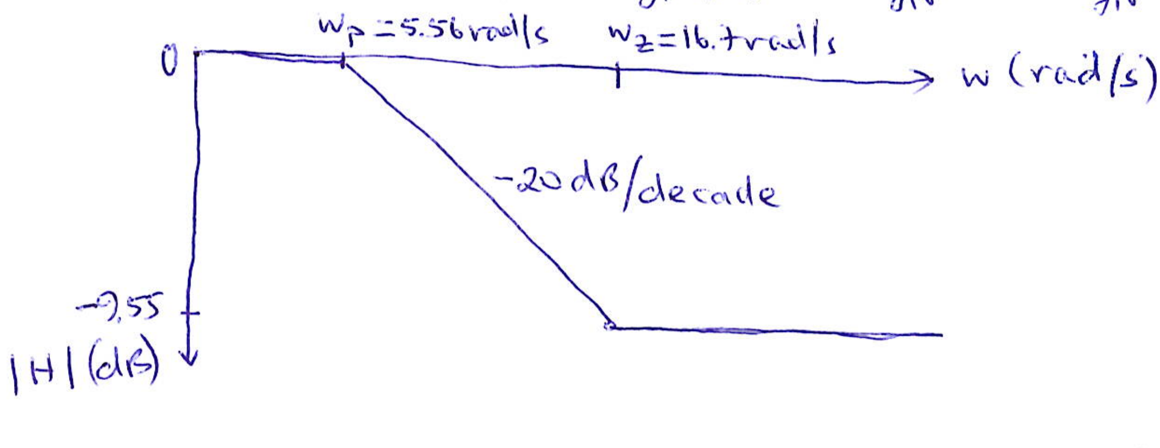
$$20\log_{10}|\bar{H}| = 20\log_{10}\sqrt{1+w^2/16.7^2} - 20\log_{10}\sqrt{1+w^2/5.56^2}$$

When $w < 5.56$ rad/s, $20\log_{10}|\bar{H}| = 20\log_{10}\sqrt{1} - 20\log_{10}\sqrt{1}$ $= 0$ dB.

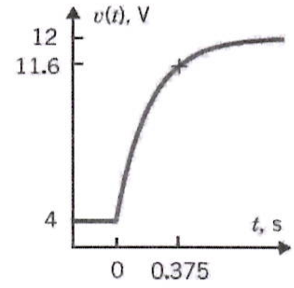
When 5.56 rad/s $< w < 16.7$ rad/s, $20\log_{10}|\bar{H}| = 0 - 20\log_{10}w + 20\log_{10}5.56$

The slope of the asymptote is -20 dB/dec. At $w=16.7$ rad/s, $20\log_{10}|\bar{H}| = 20\log_{10}5.56 - 20\log_{10}16.7 = 14.9 - 24.45 = -9.55$ dB.

When $w > 16.7$ rad/s, $20\log_{10}|\bar{H}| = 20\log_{10}5.56 - 20\log_{10}16.7 = -9.55$ dB



(20 pts) **Q.3** Given that $\mathcal{L}[v(t)] = \frac{as+b}{s^2+8s}$ where $v(t)$ is the voltage shown in the following figure, determine the values of a and b.



From the graph,

$v(0) = 4$ and $\lim_{t\to\infty} v(t) = 12$

Using the final value theorem, we get

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0} s \cdot \frac{as+b}{s^2+8s}$$

$$= \lim_{s\to 0} \frac{as+b}{s+8} = \frac{b}{8}$$

Since $\lim_{t\to\infty} v(t) = 12$, $\frac{b}{8} = 12 \Rightarrow b = 12 \times 8 = 96$

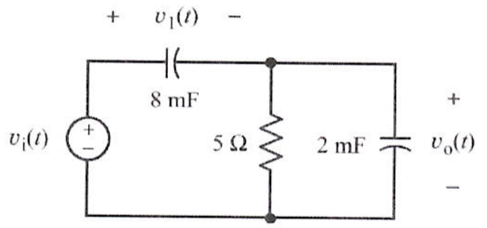
Similarly, using the initial value theorem, we obtain

$$\lim_{t\to 0} v(t) = \lim_{s\to\infty} sV(s) = \lim_{s\to\infty} s\frac{as+b}{s^2+8s} = a$$

$v(0) = 4 \Rightarrow a = 4$

(20 pts) **Q.4** The input to the circuit shown in the following figure is the voltage of voltage source

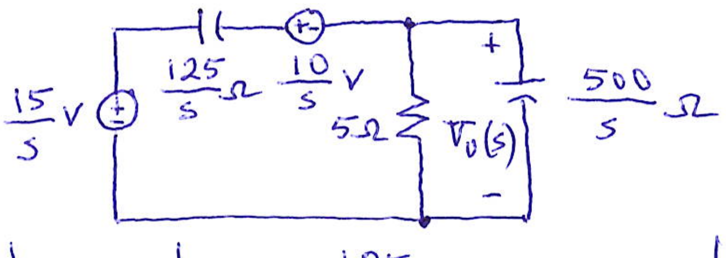
$$v_i(t) = 10 + 5u(t)\text{ V} = \begin{cases} 10\text{ V} & \text{when } t < 0 \\ 15\text{V} & \text{when } t > 0 \end{cases}$$



Determine the response $v_o(t)$.

The circuit is at steady state when $t<0$. Therefore, the capacitors act like open circuits. Since the current in the resistor is 0 A, $v_o(0^-) = 0$ V. Then KVL gives $v_i(0^-) = 10$ V.

Using the initial conditions to represent the circuit for $t>0$ in the s-domain, we have the following circuit:



$$Z_{c_1} = \frac{1}{C_1 s} = \frac{1}{8\times10^{-3} s} = \frac{125}{s}\,\Omega, \quad Z_{c_2} = \frac{1}{2\times10^{-3} s} = \frac{500}{s}\,\Omega$$

$$\frac{v_i(0^-)}{s} = \frac{10}{s}\text{ V.}$$

By the voltage division rule,

$$V_o(s) = \frac{5 \,\|\, \frac{500}{s}}{\frac{125}{s} + 5\,\|\,\frac{500}{s}}\left(\frac{15}{s} - \frac{10}{s}\right) = \frac{\frac{5\times500/s}{5+500/s}}{\frac{125}{s} + \frac{5\times500/s}{5+500/s}}\left(\frac{5}{s}\right)$$

$$= \frac{0.8s}{s+20}\cdot\frac{5}{s} = \frac{4}{s+20}$$

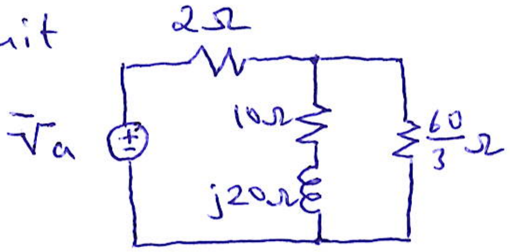
Taking the inverse Laplace transform of $V_o(s)$ gives $v_o(t) = 4e^{-20t}$ V when $t>0$.

(20 pts) **Q.5** A three-phase source with a line voltage of 45 kV rms is connected to two balanced loads. The Y-connected load has $\mathbf{Z} = 10 + j20\ \Omega$ and the Δ load has a branch impedance of 60 Ω. The connecting lines have an impedance 2 Ω. Determine the power delivered to the loads and the power lost in the wires.

Convert the delta load to an equivalent Y-connected load:

$$Z_\Delta = 60\,\Omega \Rightarrow Z_Y = \frac{Z_\Delta}{3} = 20\,\Omega$$

The per-phase equivalent circuit is shown to the right.



The phase voltage of the source is

$$\bar{V}_a = \frac{45}{\sqrt{3}}\angle 0^\circ = 26\angle 0^\circ \text{ kV rms}$$

The equivalent impedance of the load together with the line is

$$Z_{eq} = (10+j20)\,\|\,20 + 2$$

$$Z_{eq} = \frac{(10+j20)20}{10+j20+20} + 2 = 12.8 + j6.2 = 14.2\angle 25.8^\circ\,\Omega$$

The line current is

$$\bar{I}_{aA} = \frac{\bar{V}_a}{Z_{eq}} = \frac{26\times10^3\angle 0^\circ}{14.2\angle 25.8^\circ} = 1831\angle -25.8^\circ \text{ A rms.}$$

The power delivered to the load is

$$P_{loads} = 3\,|\bar{I}_{aA}|^2 \times Re[(10+j20)\,\|\,20] = 3\times 1831^2 \times 10.8$$
$$= 1.086\times10^8 = 108.6\times10^6\text{ W} = 108.6\text{ MW}$$

The power lost in the line is

Useful Info:

$$P_{line} = 3\times|\bar{I}_{aA}|^2 \times Re[Z_{line}] = 3\times 1831^2\times 2 = 20.1\times10^6\text{ W}$$
$$= 20.1\text{ MW}$$

Phasors:

$\mathbf{V}(\omega) = V_m e^{j\theta_v} = V_m\angle\theta_v$ $\quad$ $\mathbf{I}(\omega) = I_m e^{j\theta_i} = I_m\angle\theta_i$ $\quad$ $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$\mathbf{V}_\mathrm{L}(\omega) = \mathrm{Z_L}\mathbf{I}_\mathrm{L}(\omega)$ $\quad$ $\mathbf{V}_\mathrm{C}(\omega) = \mathrm{Z_C}\mathbf{I}_\mathrm{C}(\omega)$ $\quad$ $\mathbf{V}_\mathrm{R}(\omega) = \mathrm{Z_R}\mathbf{I}_\mathrm{R}(\omega)$

$\mathrm{Z_L} = j\omega L$ $\quad$ $\mathrm{Z_C} = 1/j\omega C$ $\quad$ $\mathrm{Z_R} = R$

Note that ω is the angular frequency, which is $2\pi f$.

The **rms** value of a periodic waveform $w(t)$ is defined as: $\quad W_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} w^2(t)dt}$

For a **sinusoidal waveform** $v(t) = V_m \cos(\omega t + \theta)$: $V_{rms} = \frac{V_m}{\sqrt{2}}$
Equivalent impedance when the impedances are in series $Z_{eq} = \sum_{i=1}^{k} Z_i = Z_1 + Z_2 + \cdots + Z_k$.
Equivalent impedance when the impedances are in parallel $\frac{1}{Z_{eq}} = \sum_{i=1}^{k} \frac{1}{Z_i} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_k}$.
The **average power** $P = V_{rms} I_{rms} \cos(\theta_V - \theta_I)$. The **reactive power** $Q = V_{rms} I_{rms} \sin(\theta_V - \theta_I)$.
The **complex power** $\mathbf{S} = \frac{\mathbf{VI}^*}{2} = \frac{V_m I_m}{2} \angle(\theta_V - \theta_I) = P + jQ$.
The **apparent power** is $|\mathbf{S}| = \frac{V_m I_m}{2}$. The **power factor** $pf = \cos(\theta_V - \theta_I)$.
For the **maximum power transfer**, $Z_L = Z_t^*$, where Z_L is the load impedance and Z_t^* is the complex conjugate of the Thévenin equivalent impedance Z_t of the circuit.
For **coupled inductors** in the time domain $v_1 = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}$ and $v_2 = L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$
For an **ideal transformer**, $\mathbf{V}_2 = \frac{N_2}{N_1} \mathbf{V}_1$ and $\mathbf{I}_1 = -\frac{N_2}{N_1} \mathbf{I}_2$.
Total complex power in a **balanced three-phase** circuits: $\boldsymbol{S}_T = 3\boldsymbol{S}_p = 3\frac{\mathbf{VI}^*}{2}$.
In **balanced three-phase** circuits, $V_{LL} = \sqrt{3} V_p$ and $I_L = I_p$ for Y connection; $V_{LL} = V_p$ and $I_L = \sqrt{3} I_p$ for Δ connection.
Network function: $\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$, where $\mathbf{X}(\omega)$ is the phasor corresponding to the input to the circuit and $\mathbf{Y}(\omega)$ is the phasor corresponding to the steady-state response of the network.
Gain: $|\mathbf{H}(\omega)| = \left|\frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}\right|$, **phase shift:** $\angle\mathbf{H}(\omega) = \angle\mathbf{Y}(\omega) - \angle\mathbf{X}(\omega)$.
Logarithmic gain: $20 \log_{10}|\mathbf{H}(\omega)|$ in decibel (**dB**).
Laplace transform: $F(s) = \int_0^\infty f(t) e^{-st}\, dt$, where s is a complex variable given by $s = \sigma + j\omega$.
For an **inductor**, its **time-domain** equation $v(t) = L \frac{d}{dt} i(t)$.
For a **capacitor**, its **time-domain** equation $i(t) = C \frac{d}{dt} v(t)$.
The **transfer function** of a circuit is defined as the ratio of the Laplace transform of the response of the circuit to the Laplace transform of the input to the circuit when the initial conditions are zero.
The **transfer function** is given by $H(s) = \frac{V_o(s)}{V_i(s)}$.
In an **ideal op-amp**: $i_+ = i_- = 0$ $v_+ = v_-$
The **fundamental frequency** ω_0 of the periodic function $f(t)$ is $\omega_0 = 2\pi/T$, where T is the period.

The **trigonometric Fourier series**: $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$

$$a_0 = \frac{1}{T}\int_{t_0}^{T+t_0} f(t)dt = \text{average value of } f(t)$$

$$a_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t) \cos n\omega_0 t\, dt, \quad n > 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t) \sin n\omega_0 t\, dt, \quad n > 0$$

Fourier transform: $F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$, where ω is the frequency.
Inverse Fourier transform: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$

Nano (**n**) → 10^{-9} Micro (**μ**) → 10^{-6} Milli (**m**) → 10^{-3} Kilo (**k**) → 10^{3} Mega (**M**) → 10^{6}

$f(t)$ for $t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

PROPERTY	$f(t),\ t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time scaling	$f(at)$, where $a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
Time differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - \left(sf(0^-) + \frac{df(0^-)}{dt}\right)$
	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}f(0^-)}{dt^{k-1}}$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time convolution	$f_1(t)^*f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$	$F_1(s)F_2(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda)d\lambda$
Frequency differentiation	$tf(t)$	$-\frac{dF(s)}{ds}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$