lecture6: t and F-distribution

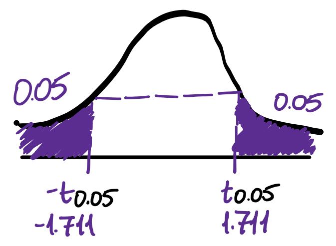
20 Mart 2025 Perşembe 11:23

t-distribution

Example 8.11: A chemical engineer claims that the population mean yield of a certain batch process is 500 grams per milliliter of raw material. To check this claim he samples 25 batches each month. If the computed t-value falls between $-t_{0.05}$ and $t_{0.05}$, he is satisfied with this claim. What conclusion should he draw from a sample that has a mean $\bar{x} = 518$ grams per milliliter and a sample standard deviation $s = 40$ grams? Assume the distribution of yields to be approximately normal.

$H_0: \mu = 500$

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{518 - 500}{\frac{40}{\sqrt{25}}} = 2.25$$



Since $t = 2.25 > t_{0.05}$, the claim is not true.

$v = n - 1 = 24$

Population I: μ_1, σ_1^2

Population II: μ_2, σ_2^2

$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \sigma^2_{\bar{X}_1 - \bar{X}_2})$$

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$var(\bar{X}_1 - \bar{X}_2) = var(\bar{X}_1) + var(\bar{X}_2) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n_2}$$

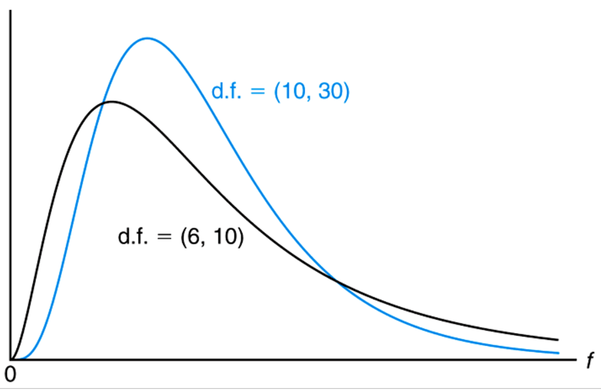
$$z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

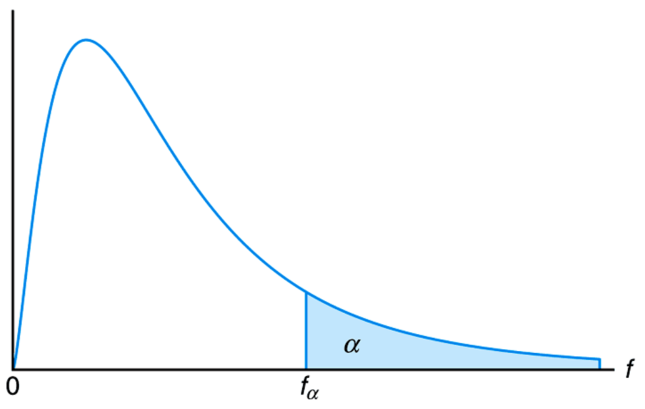
F-distribution

Let U and V be two independent random variables having chi-squared distributions with v_1 and v_2 degrees of freedom, respectively. Then the distribution of the random variable $F = \frac{U/v_1}{V/v_2}$ is given by the density function

$$h(f) = \begin{cases} \frac{\Gamma[(v_1+v_2)/2](v_1/v_2)^{v_1/2}}{\Gamma(v_1/2)\Gamma(v_2/2)} \frac{f^{(v_1/2)-1}}{(1+v_1 f/v_2)^{(v_1+v_2)/2}}, & f > 0, \\ 0, & f \le 0. \end{cases}$$

This is known as the **F-distribution** with v_1 and v_2 degrees of freedom (d.f.).





Writing $f_\alpha(v_1, v_2)$ for f_α with v_1 and v_2 degrees of freedom, we obtain

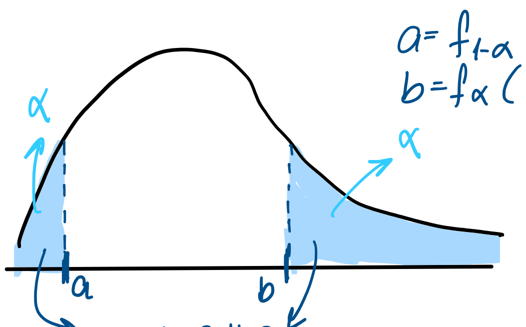
$$f_{1-\alpha}(v_1, v_2) = \frac{1}{f_\alpha(v_2, v_1)}.$$

therefore, we can find $f_{0.95}$ from $\alpha = 0.05$ Table

$$f_{0.95}(8,9) = \frac{1}{f_{0.05}(9,8)} = \frac{1}{3.39}$$

$a = f_{1-\alpha}(v_1, v_2)$

$b = f_\alpha(v_1, v_2)$



Cutoff points for the F Distribution (α = 0.05)

v_2 →, v_1 ↓

	1	2	3	4	5	6	7	8	9	10	12	15
1	161,4	199,5	215,7	224,6	230,2	234,0	236,8	238,9	240,5	241,9	243,9	245,9
2	18,51	19,00	19,16	19,25	19,30	19,33	19,35	19,37	19,38	19,40	19,41	19,43
3	10,13	9,55	9,28	9,12	9,01	8,94	8,89	8,85	8,81	8,79	8,74	8,70
4	7,71	6,94	6,59	6,39	6,26	6,16	6,09	6,04	6,00	5,96	5,91	5,86
5	6,61	5,79	5,41	5,19	5,05	4,95	4,88	4,82	4,77	4,74	4,68	4,62
6	5,99	5,14	4,76	4,53	4,39	4,28	4,21	4,15	4,10	4,06	4,00	3,94
7	5,59	4,74	4,35	4,12	3,97	3,87	3,79	3,73	3,68	3,64	3,57	3,51
8	5,32	4,46	4,07	3,84	3,69	3,58	3,50	3,44	3,39	3,35	3,28	3,22
9	5,12	4,26	3,86	3,63	3,48	3,37	3,29	3,23	3,18	3,14	3,07	3,01

If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 taken from normal populations with variances σ_1^2 and σ_2^2, respectively, then

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$$

has an F-distribution with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom.

Example

Let S_1^2 denote the sample variance for a random sample of size 10 from Population I and let S_2^2 denote the sample variance for a random sample of size 8 from Population II. The variance of Population I is assumed to be three times the variance of Population II. Find two numbers a and b such that $P(a \le S_1^2/S_2^2 \le b) = 0.90$ assuming S_1^2 to be independent of S_2^2.

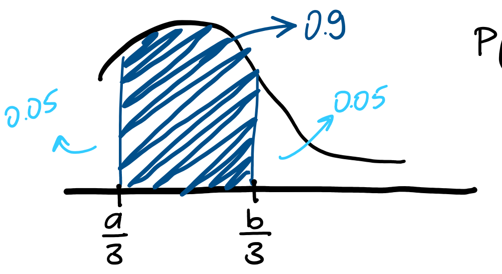
Population I: $n = 10$, σ_1^2, S_1^2

Population II: $n = 8$, $\sigma_1^2 = 3\sigma_2^2$, S_2^2

$$F = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} = \frac{1}{3} \frac{S_1^2}{S_2^2}$$

$v_1 = n_1 - 1 = 10 - 1 = 9$

$v_2 = n_2 - 1 = 8 - 1 = 7$



$$P\left(\frac{a}{3} \le \frac{S^2}{\sigma_2^2} \cdot \frac{1}{3} \le \frac{b}{3}\right) = 0.9$$

$$P\left(\frac{a}{3} \le F \le \frac{b}{3}\right) = 0.9$$

$-f_{0.05}(9,7) = \frac{a}{3}$

$f_{0.05}(9,7) = \frac{b}{3}$

3.68

$3 \times 3.68 = b \rightsquigarrow b = 11.04$

$$\frac{1}{3.29} = \frac{1}{f_{0.05}(7,9)} = f_{0.95}(9,7) = \frac{a}{3}$$

$a = 0.91$

Cutoff points for the F Distribution (α = 0.01)

	1	2	3	4	5	6	7	8	9	10	12	15
1	4052	5000	5403	5625	5764	5859	5928	5981	6022	6056	6106	6157
2	98,50	99,00	99,17	99,25	99,30	99,33	99,36	99,37	99,39	99,40	99,42	99,43
3	34,12	30,82	29,46	28,71	28,24	27,91	27,67	27,49	27,35	27,23	27,05	26,87
4	21,20	18,00	16,69	15,98	15,52	15,21	14,98	14,80	14,66	14,55	14,37	14,20
5	16,26	13,27	12,06	11,39	10,97	10,67	10,46	10,29	10,16	10,05	9,89	9,72
6	13,75	10,92	9,78	9,15	8,75	8,47	8,26	8,10	7,98	7,87	7,72	7,56
7	12,25	9,55	8,45	7,85	7,46	7,19	6,99	6,84	6,72	6,62	6,47	6,31
8	11,26	8,65	7,59	7,01	6,63	6,37	6,18	6,03	5,91	5,81	5,67	5,52
9	10,56	8,02	6,99	6,42	6,06	5,80	5,61	5,47	5,35	5,26	5,11	4,96
10	10,04	7,56	6,55	5,99	5,64	5,39	5,20	5,06	4,94	4,85	4,71	4,56
11	9,65	7,21	6,22	5,67	5,32	5,07	4,89	4,74	4,63	4,54	4,40	4,25
12	9,33	6,93	5,95	5,41	5,06	4,82	4,64	4,50	4,39	4,30	4,16	4,01

$\alpha = P(F > 4.89) = f_\alpha(7, 11)$

Example

If S_1^2 and S_2^2 represent the variances of independent random samples of size $n_1 = 8$ and $n_2 = 12$, taken from normal populations with equal variances, find $P(S_1^2/S_2^2 < 4.89)$.

$v_1 = 7$, $n_1 = 8$; $v_2 = 11$, $n_2 = 12$

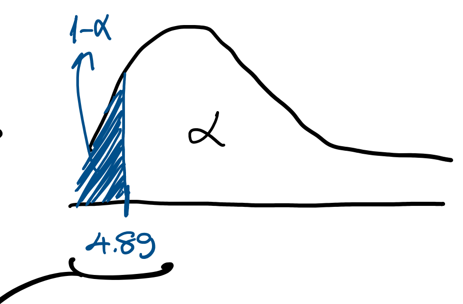
$$P\left(\frac{S_1^2}{S_2^2} < 4.89\right) = ?$$

$\sigma_1^2 = \sigma_2^2$

$$F = \frac{\cancel{\sigma_2^2} S_1^2}{\cancel{\sigma_1^2} S_2^2} = \frac{S_1^2}{S_2^2}$$

since equal

$$P(F_{v_1, v_2} < 4.89) = ?$$



since this value is exactly at $\alpha = 0.01$ table, we use that table

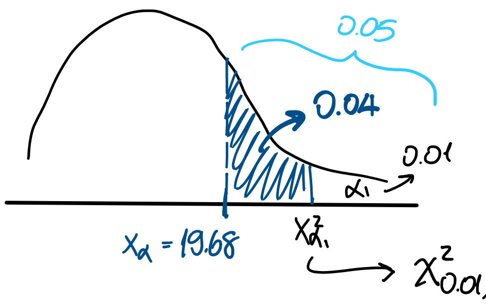
$F > 4.89 \Rightarrow f_\alpha(7,11)$ $\alpha = 0.01$

$f_{1-\alpha}(7,11) \Rightarrow \alpha = 0.99$

example (chi-square)

For a chi-square distribution, find χ^2_α such that

* $P(19.68 < \chi^2 < \chi^2_\alpha) = 0.04$, when $v = 11$



$v = 11$ $\chi^2_{\alpha, 11} = 19.68$

$\alpha = 0.05$

$\chi^2_{0.01, 11} = 24.725$